Chapter 2. Elementary coordinate geometry

Using a computer to produce pictures requires translating geometry to numbers and *vice-versa*. Carrying out these tasks, of course, requires a coordinate system. Through nearly all of this course, the coordinate systems we use will have the property that the x and y axes are perpendicular to each other and measured in the same units. PostScript can in fact use any linear coordinate system, but strange effects may be generated.

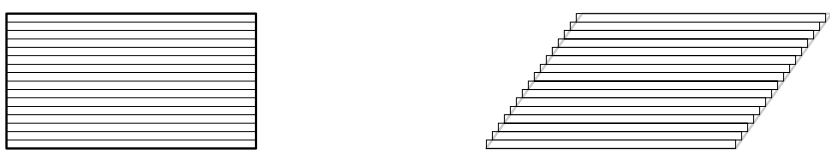
The first topic will recall how to calculate lengths in such a coordinate system, which relies simply on Pythagoras' Theorem. I shall recall in the text a single proof of this, and suggest others in exercises. Of course many proofs are known—the one I present is a variant of Euclid's. It requires a preliminary discussion of shears.

1. Shears

A shear is a transformation of a $2D$ figure that has this effect:

It is a bit hard to describe in plain language. In fact, as a secondary school teacher once pointed out to me, it is hard to illustrate by any normal means, since it really requires tricky animation to give the right impression. To illustrate other transformations like translations or rotations you can move an object (for example a sheet of paper) around in front of yourself, but shears are not easily simulated.

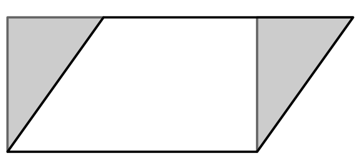
A shear can perhaps be best illustrated by thinking of the rectangle as a side view of a deck of thick cards:



In other words it slides the components of a figure past each other, and it slides things further if they are higher. From this picture it should be at least intuitively clear that

- *Shears preserve area.*

Roughly speaking this is because sliding a very thin piece of a figure doesn't change its shape, much less its area. The actual proof that a shear doesn't change area is also very simple, at least if the shear has small enough effect:



The idea is that we lop off a triangle from one end and shift it around to the other in order to make a parallelogram into a rectangle. The reason this works is because we shift that triangle without distorting it. If the shear is a large one, then it can be expressed as a sequence of small ones applied one after the other, and hence still preserves area.

That shears preserve area is equivalent to the familiar claim that the area of a parallelogram is equal to base $\times$ height.

Of course we have to appeal to some more fundamental result to justify this argument. A rigourous proof can be put together by discussing angles cut off by parallel lines. Ultimately it derives from Euclid's parallel postulate, but I won't discuss this further.

Exercise 1.1. *Read Euclid's proof of his proposition I.35. Reproduce diagrams illustrating his proof in PostScript.*

Exercise 1.2. *Picturing the rectangle as a stack of cards can lead to a valid proof that shears preserve area. Find it, illustrate it.*

Exercise 1.3. *The simplest way to tell that two regions in the plane have the same area is to decompose each of them into smaller regions which match perfectly (are congruent to) the regions in the other, but possibly rearranged. For example, the first picture above shows precisely in this way that the rectangle and the parallelogram have the same area.*

Can this be done for all parallelograms? In other words, if we are given two parallelograms of the same area, can you give an algorithm to dissect them into matching congruent regions? Show some simple examples first. For example, you might suppose both the parallelograms to be rectangles, and even one of them to be a square. (Warning: this is not too difficult, but not quite trivial either.)

Exercise 1.4. *Explain how to prove that shears preserve area by exhibiting congruent dissections of a parallelogram and a rectangle with the same base and height. (Warning: exactly how this works will depend on the particular parallelogram. Experiment.)*

2. Lengths

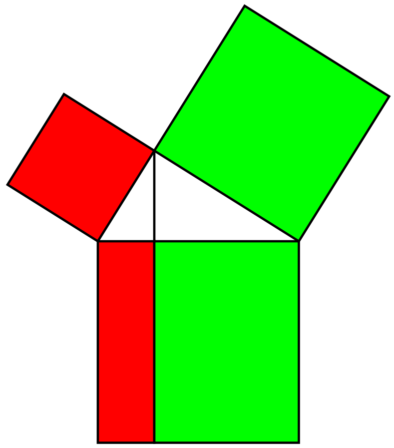
We begin with the statement of Pythagoras' Theorem:

- *For a right triangle with short sides a and b and long side c we have $c^2 = a^2 + b^2$.*

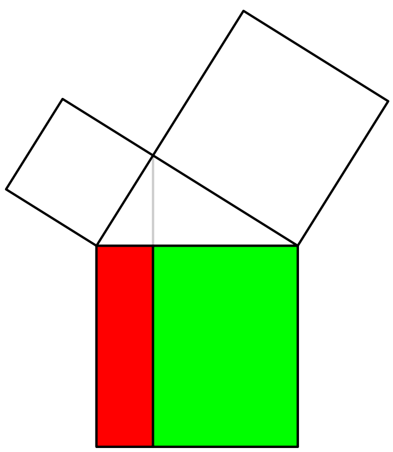
For the coordinate systems we are working with, those in which x and y are measured uniformly and the x and y axes are perpendicular to each other, this has as consequence that

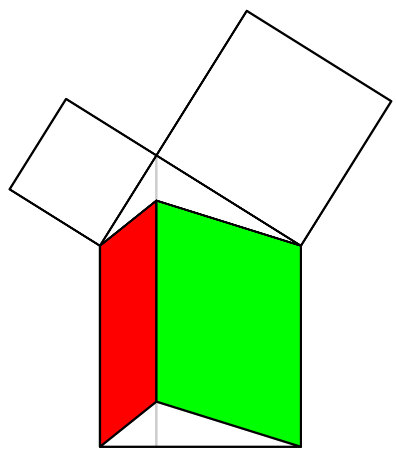
- *The distance from the origin to (x, y) is $\sqrt{x^2 + y^2}$.*

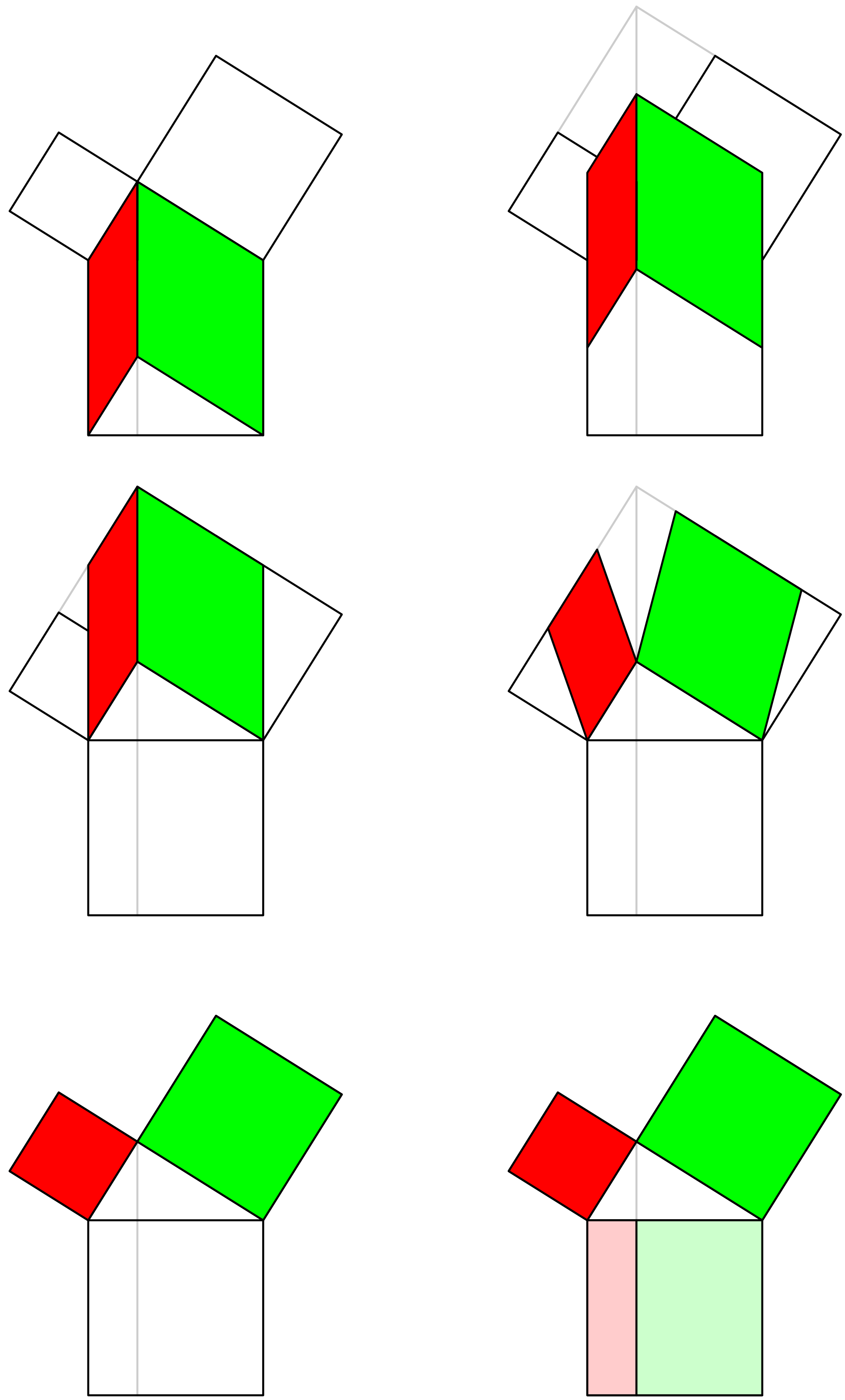
Nearly everyone sees a proof of this in secondary school, but the proofs are rarely memorable. The point of our proof (and Euclid's) is that one can explicitly decompose the large square into two rectangles, each of which matches one of the smaller squares in area. We do this by dropping a perpendicular from the right angle vertex across to the hypotenuse and through to the base of the large square.



The similarly coloured regions have equal areas. The proof (which I learned from Howard Eves, and which perhaps derives from the classical proof of Pappus' Theorem) proceeds by performing a series of shears and translations, which are area-preserving, to transform the rectangles into the corresponding squares.







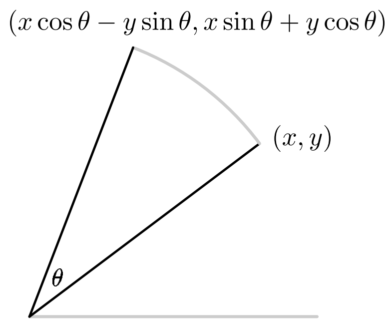
3. Rotations

Suppose we rotate the point in the plane with coordinates (x, y) around the origin through an angle of θ. What are the coordinates of the point we then get? The answer is

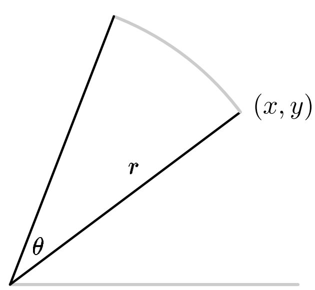
- *If we rotate the point (x, y) around the origin through angle θ, the point we get is*

$$(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) .$$

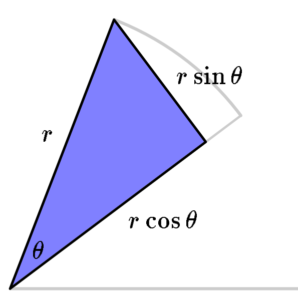
Here's how it looks:



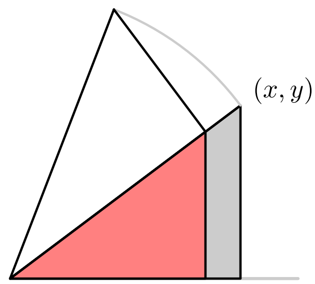
This is a formula used repeatedly, in various guises and in many different circumstances throughout these notes. The proof to be given here is very direct.

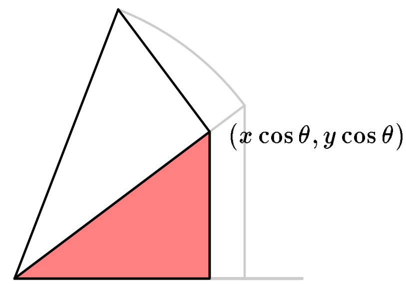


The first step is to drop a perpendicular from the rotated point onto the radius vector of (x, y). Elementary trigonometry gives us the dimensions of the triangle we get.



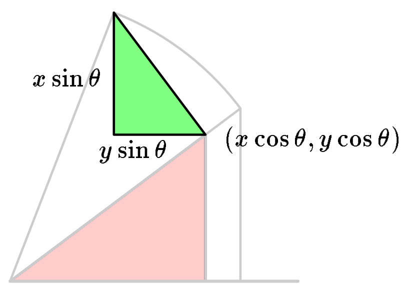
What are the coordinates of the bottom of the perpendicular? Because we have rotated through an angle of θ and rotation preserves distances from the origin, the distance from the origin to the bottom of the perpendicular is $r\cos\theta$, where $r = \sqrt{x^2+y^2}$ is the length of the original vector (x, y). The length of the perpendicular itself is $r\sin\theta$.





Since the triangle on the right is obtained from the one on the left by a simple scaling operation they are similar. The ratio of the long sides is $r\cos\theta : r$, so the coordinates of the bottom of the perpendicular are $(x\cos\theta, y\cos\theta)$.

We now add a triangle to the picture:



It also is similar to the one of the triangles in the previous figure (the angle at its lower right is obtained by a simple rotation from one of the angles in the smaller of those two), and since its long side is $r\sin\theta$ its bottom has length $x\sin\theta$ and the left side length $y\sin\theta$.

But this tells us immediately that the x-coordinate of the rotated point is $x\cos\theta - y\sin\theta$, and its y-coordinate is $y\cos\theta + x\sin\theta$.

If we take (x, y) to be the point $(\cos\varphi, \sin\varphi)$ we get by rotating $(1, 0)$ through an angle of φ, then on the one hand we get the vector $(\cos(\varphi+\theta), \sin(\varphi+\theta))$ that we would get by rotating $(1, 0)$ through an angle of $\varphi+\theta$, and on the other the formula we have just proven gives a different expression. Therefore

$$(\cos(\varphi+\theta), \sin(\varphi+\theta)) = (\cos\varphi\cos\theta - \sin\varphi\sin\theta, \cos\varphi\sin\theta + \sin\varphi\cos\theta)$$

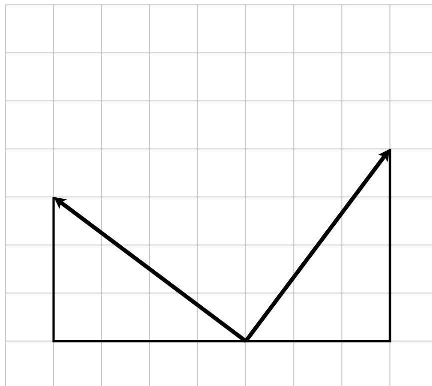
In fact, the rotation formula is equivalent to the pair of trigonometrical formulas

$$\begin{aligned}\cos(\varphi+\theta) &= \cos\varphi\cos\theta - \sin\varphi\sin\theta \\ \sin(\varphi+\theta) &= \sin\varphi\cos\theta + \cos\varphi\sin\theta\end{aligned}$$

There is one simple case of the rotation formula which is used very often.

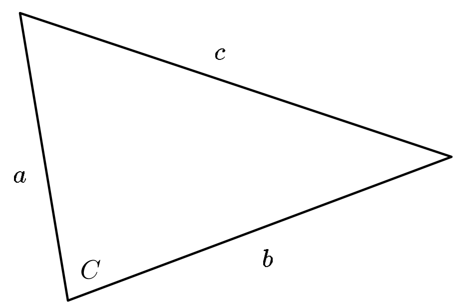
- *If (x, y) are the coordinates of a vector then $(-y, x)$ are the coordinates of the vector rotated through a right angle in the positive direction.*

This can be seen directly:



4. Angles

The **cosine rule** is a generalization of Pythagoras' Theorem which applies to triangles not necessarily possessing a right angle.

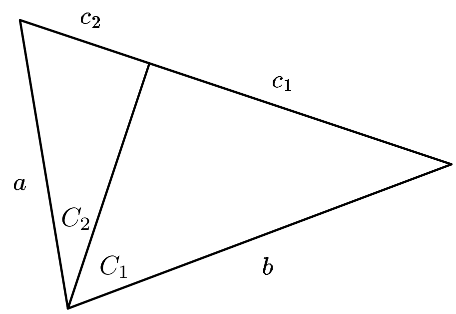


- *In a triangle with sides a, b, c and angle C opposite c*

$$c^2 = a^2 + b^2 - 2ab\cos C \ .$$

I shall give two proofs.

The first is more algebraic, and depends on the cosine sum formula.



Let the side opposite the origin have length c. By 'dropping' a perpendicular from the origin onto this side we decompose it into two pieces of length, say, c_1 and c_2. Thus

$$c^2 = (c_1 + c_2)^2 = c_1^2 + c_2^2 + 2c_1c_2$$

On the other hand the original angle θ is decomposed into two parts C_1, C_2. We know that

$$\cos C = \cos C_1 \cos C_2 - \sin C_1 \sin C_2$$

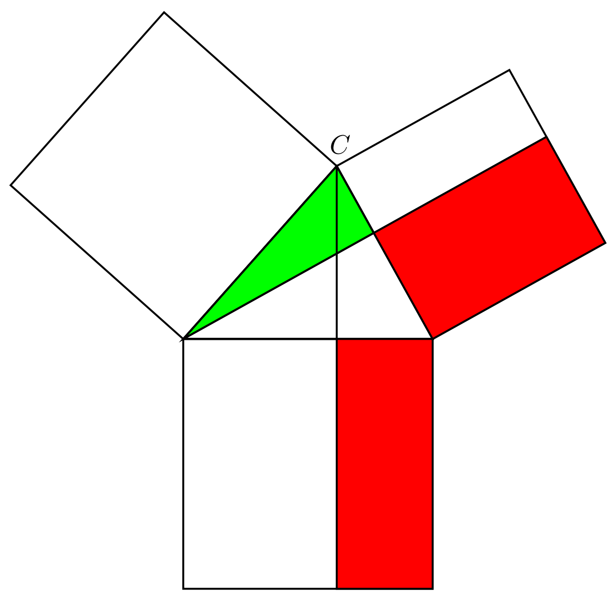
Finally, let y be the length of the perpendicular. By Pythagoras' Theorem applied to each of the small triangles and trigonometry

$$\begin{aligned} c_1^2 &= a^2 - y^2 \\ c_2^2 &= b^2 - y^2 \\ c_1 &= a \sin C_1 \\ y &= a \cos C_1 \\ c_2 &= a \sin C_2 \\ y &= a \cos C_2 \end{aligned}$$

so that

$$\begin{aligned} c^2 &= (a^2 - y^2) + (b^2 - y^2) + 2ab \sin C_1 \sin C_2 \\ &= a^2 + b^2 - 2ab\, \cos C_1 \cos C_2 + 2ab \sin C_1 \sin C_2 \\ &= a^2 + b^2 - 2ab \cos C\ . \end{aligned}$$

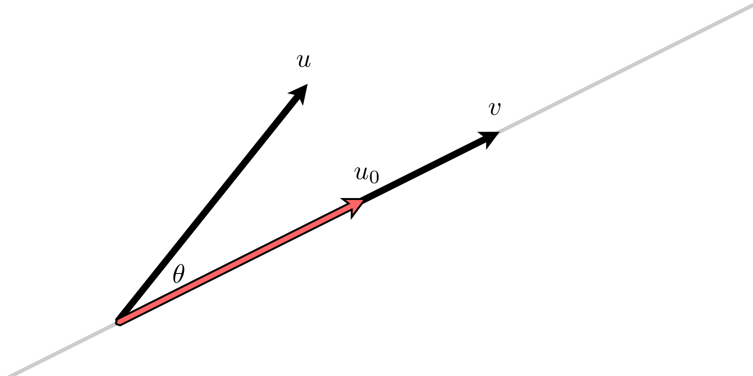
The second generalizes Euclid's proof of Pythagoras' Theorem.



Exercise 4.1. *Follow hints in the above figure to prove the cosine rule. Use a sequence of colour drawings, made either by PostScript or by hand. (Hint: show that the two rectangles have the same area.)*

5. Perpendicular projections

Suppose u and v to be vectors, and ℓ the line along v. Let u_0 be the perpendicular projection of u onto ℓ.

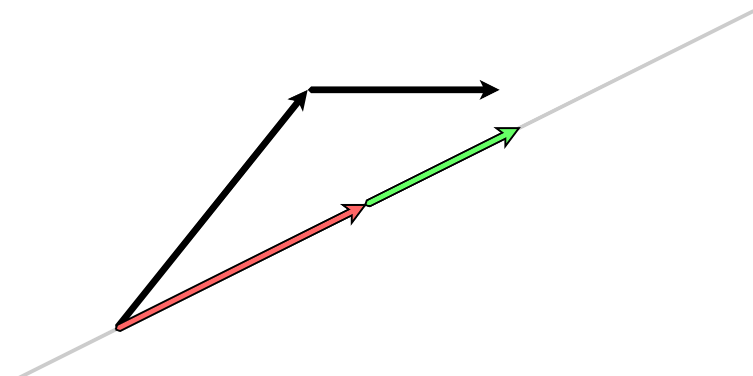


On the one hand the length of the projection u_0 is $\|u\|\,|\cos\theta|$. Define the **algebraic projection length** of u to be this length if u_0 has the same direction as v, otherwise its negative. The signs work out exactly so that the algebraic projection length is in fact

$$\|u\|\,\cos\theta\ .$$

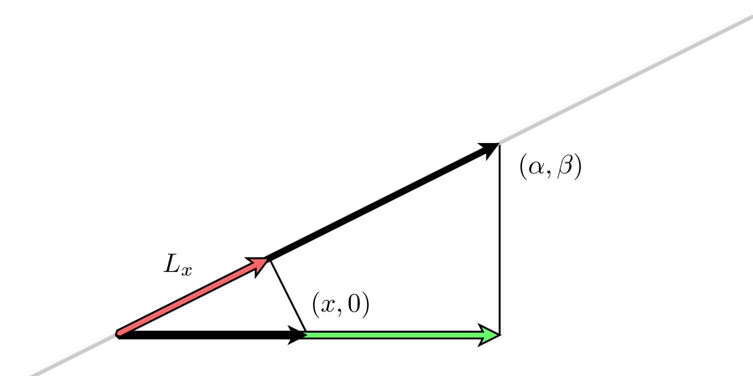
We shall now derive a formula for the algebraic projection length in terms of the coordinates of u and v.

The first observation is that the algebraic projection length is an additive function of u, which means that if $u = u_1 + u_2$ then the algebraic projection length of u is the sum of the lengths for u_1 and u_2.



Since u is equal to the sum of its projections onto the x and y axes, it is only necessary to find the algebraic projection lengths of $(x, 0)$ and $(0, y)$ and add them together.

Let's look at $(x, 0)$.



Let L_x be the algebraic projection length of $(x, 0)$, and let $v = (\alpha, \beta)$. The two triangles are similar, so we see that

$$\frac{L_x}{x} = \frac{\alpha}{\|v\|} \ .$$

Similarly if L_y is that of $(0, y)$ then

$$\frac{L_y}{y} = \frac{\beta}{\|v\|} \ .$$

Hence the algebraic projection length of $u = (x, y)$ is

$$L_x + L_y = \frac{\alpha x + \beta y}{\|v\|} \ .$$

If we combine the two formulas we have for the algebraic projection length we get a formula for $\cos\theta$: and finally

$$\|u\| \, \cos\theta = \frac{\alpha x + \beta y}{\|v\|}, \qquad \cos\theta = \frac{\alpha x + \beta y}{\|u\| \, \|v\|} \ .$$

6. Dot products

If $u = (x, y)$ and $v = (\alpha, \beta)$ then their **dot product** is defined to the the left hand side of the formula above:

$$u \bullet v = \alpha x + \beta y \ .$$

We have seen that the dot product can be interpreted geometrically in terms of the angle between them.

The dot product of two vectors in any number n of dimensions is **by definition** the sum of the products of their coordinates:

$$(x_1, x_2, \ldots, x_n) \bullet (y_1, y_2, \ldots, y_n) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \ .$$

There are a number of simple formal algebraic rules it satisfies:

$$\begin{aligned} cx \bullet y &= c(x \bullet y) \\ x \bullet cy &= c(x \bullet y) \\ (x + y) \bullet z &= x \bullet z + y \bullet z \\ x \bullet x &= \|x\|^2 \end{aligned}$$

where $\|x\|$ is the length of the vector x, the distance of its head from its tail. What we have seen to be true in two dimensions holds also in three dimensions:

- *For vectors u and v in 2 or 3 dimensions*

$$u \bullet v = \|u\| \, \|v\| \, \cos\theta$$

where θ is the angle between u and v.

We can prove this geometrically as we did in $2D$, but we can also proceed more algebraically, applyng the cosine rule. If u and v are vectors, then they form two sides of a triangle. On the one hand, the square of the length of the third side is

$$\|u - v\|^2 = \|u - v\| \bullet \|u - v\| = \|u\|^2 + \|v\|^2 - 2u \bullet v$$

and on the other, by the cosine rule, it is

$$\|u\|^2 + \|v\|^2 - 2\|u\|\,\|v\| \cos\theta$$

so by comparison

$$u \bullet v = \|u\|\,\|v\| \cos\theta\ .$$

In particular:

- *The dot product of two vectors is* 0 *precisely when they are perpendicular to each other.*

Of course it is only in 2 or 3 dimensions that we have a geometric definition of the angle between two vectors. In higher dimensions, this formula is used to define that angle algebraically.

7. Lines

There are several ways to determine lines in the plane.

- *A line is determined by a pair of distinct points* P *and* Q.
- *A line is determined by its equation*

$$Ax + By = C\ .$$

  As a special case of this we have the slope-intercept form

$$y = mx + b$$

  which describes lines which are not dead vertical.
- *A line is determined by a point* P *and a direction away from that point. If* $v = (\Delta x, \Delta y)$ *is a vector in that direction then the line is the set of points*

$$P + tv$$

  where t *ranges over all real numbers. This is called the* **parametric representation** *of the line.*

In this section we shall answer two questions. The first is

(1) *What is the geometrical significance of the equation*

$$Ax + By = C?$$

The equation

$$Ax + By = C$$

can be rewritten as

$$(A, B) \bullet (x, y) = C\ .$$

which means that the corresponding line is that of all vectors (x, y) have a fixed dot product with (A, B). If (x_0, y_0) and (x_1, y_1) are two points on the same line then

$$(A, B) \bullet (x_0, y_0) = (A, B) \bullet (x_1, y_1), \qquad (A, B) \bullet (x_0 - y_1, y_0 - y_1) = 0$$

which means that their difference is is perpendicular to (A, B). In other words, the direction of the line is perpendicular to (A, B).

There will be exactly one vector (x, y) on the line which is a multiple of (A, B), say $t(A, B)$. We can solve:

$$(A, B) \bullet (tA, tB) = C$$
$$t = \frac{C}{A^2 + B^2}$$

The sign of t will be the sign of C. The length of $t(A, B)$ will be

$$\left| \frac{C}{\sqrt{A^2 + B^2}} \right| .$$

In summary:

- *The vector (A, B) is perpendicular to the line $Ax + By = C$.*
- *The signed distance from the origin to this line is*

$$\frac{C}{\sqrt{A^2 + B^2}} .$$

The second:

(2) *If we are given two lines in parametric form, how do we calculate their intersection?*

The intersection of two lines in parametric form

$$\{P + tu\}, \quad \{Q + tv\}$$

is a point R which satisfies

$$R = P + tu = Q + sv$$

for some numbers t, s. Thus if w is any vector perpendicular to v (for example, v rotated by $90°$)

$$(P - Q) + tu = sv$$
$$(P - Q + tu) \bullet w = 0$$
$$t = \frac{(Q - P) \bullet w}{u \bullet w} .$$

Exercise 7.1. *Given a line $Ax + By = C$ and a point $P = (x_0, y_0)$, find a formula for the perpendicular projection of P onto the line.*

Exercise 7.2. *Given two lines $A_0x + B_0y = C_0$ and $A_1x + B_1y = C_1$, find a formula for their point of intersection.*

Exercise 7.3. *Given two points P and Q, find a formula $Ax + By = C$ for the line through them.*