On the Notion of an Automorphic Representation *

The irreducible representations of a reductive group over a local field can be obtained from the square-integrable representations of Levi factors of parabolic subgroups by induction and formation of subquotients [2], [4]. Over a global field F the same process leads from the cuspidal representations, which are analogues of square-integrable representations, to all automorphic representations.

Suppose P is a parabolic subgroup of G with Levi factor M and $\sigma = \otimes\sigma_v$ a cuspidal representation of $M(\mathbf{A})$. Then $\mathrm{Ind}\,\sigma = \otimes_v \mathrm{Ind}\,\sigma_v$ is a representation of $G(\mathbf{A})$ which may not be irreducible, and may not even have a finite composition series. As usual an irreducible subquotient of this representation is said to be a constituent of it.

For almost all v, $\mathrm{Ind}\,\sigma_v$ has exactly one constituent π_v° containing the trivial representation of $G(\mathbf{O}_v)$. If $\mathrm{Ind}\,\sigma_v$ acts on X_v then π_v° can be obtained by taking the smallest $G(F_v)$-invariant subspace V_v of X_v containing nonzero vectors fixed by $G(\mathbf{O}_v)$ together with the largest $G(F_v)$-invariant subspace U_v of V_v containing no such vectors and then letting $G(F_v)$ act on V_v/U_v.

Lemma 1. *The constituents of* $\mathrm{Ind}\,\sigma$ *are the representations* $\pi = \otimes\pi_v$ *where* π_v *is a constituent of* $\mathrm{Ind}\,\sigma_v$ *and* $\pi_v = \pi_v^\circ$ *for almost all* v.

That any such representation is a constituent is clear. Conversely let the constituent π act on V/U with $0 \subseteq U \subseteq V \subseteq X = \otimes X_v$. Recall that to construct the tensor product one chooses a finite set of places S_0 and for each v not in S_0 a vector x_v° which is not zero and is fixed by $G(\mathbf{O}_v)$. We can find a finite set of places S, containing S_0, and a vector x_S in $X_S = \otimes_{v\in S} X_v$ which are such that $x = x_S \otimes (\otimes_{v\notin S} x_v^\circ)$ lies in V but not in U.

Let V_S be the smallest subspace of X_S containing x_S and invariant under $G_S = \Pi_{v\in S} G(F_v)$. There is clearly a surjective map

$$V_S \otimes (\otimes_{v\notin S} V_v) \to V/U,$$

and if $v_0 \notin S$ the kernel contains $V_S \otimes U_{v_0} \otimes (\otimes_{v\notin S\cup\{v_0\}} V_v)$. We obtain a surjection $V_S \otimes (\otimes_{v\in S} V_v/U_v) \to V/U$ with a kernel of the form $U_S \otimes (\otimes_{v\in S} V_v/U_v)$, U_S lying in V_S. The representation of G_S on V_S/U_S is irreducible and, since $\mathrm{Ind}\,\sigma_v$ has a finite composition series, of the form $\otimes_{v\in S}\pi_v$, π_v being a constituent of $\mathrm{Ind}\,\sigma_v$. Thus $\pi = \otimes\pi_v$ with $\pi_v = \pi_v^\circ$ when $v \notin S$.

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The purpose of this note is to establish the following proposition.*

Proposition 2. *A representation π of $G(\mathbf{A})$ is an automorphic representation if and only if π is a constituent of $\mathrm{Ind}\,\sigma$ for some P and some σ.*

The proof that every constituent of $\mathrm{Ind}\,\sigma$ is an automorphic representation will invoke the theory of Eisenstein series, which has been fully developed only when the global field F has characteristic zero [3]. One can expect however that the analogous theory for global fields of positive characteristic will appear shortly; so there is little point in making the restriction to characteristic zero explicit in the proposition. Besides, the proof that every automorphic representation is a constituent of some $\mathrm{Ind}\,\sigma$ does not involve the theory of Eisenstein series in any serious way.

We begin by remarking some simple lemmas.

Lemma 3. *Let Z be the center of G. Then an automorphic form is $Z(\mathbf{A})$-finite.*

This is verified in [1].

Lemma 4. *Suppose K is a maximal compact subgroup of $G(\mathbf{A})$ and φ an automorphic form with respect to K. Let P be a parabolic subgroup of G. Choose $g \in G(\mathbf{A})$ and let K' be a maximal compact subgroup of $M(\mathbf{A})$ containing the projection of $gKg^{-1} \cap P(\mathbf{A})$ on $M(\mathbf{A})$. Then*

$$\varphi_P(m;g) = \int_{N(F)\backslash N(\mathbf{A})} \varphi(nmg)dn$$

is an automorphic form on $M(\mathbf{A})$ with respect to K'.

It is clear that the growth conditions are hereditary and that $\varphi_P(\cdot;g)$ is smooth and K'-finite. That it transforms under admissible representations of the local Hecke algebras of M is a consequence of theorems in [2] and [4].

We say that φ is orthogonal to cusp forms if $\int_{\Omega G_{\mathrm{der}}(\mathbf{A})} \varphi(g)\psi(g)\,dg = 0$ whenever ψ is a cusp form and Ω is a compact set in $G(\mathbf{A})$. If P is a parabolic subgroup we write $\psi \perp P$ if $\varphi_P(\cdot;g)$ is orthogonal to cusp forms on $M(\mathbf{A})$ for all g. We recall a simple lemma [3].

* The definition of an automorphic representation is given in the paper [1] by A. Borel and H. Jacquet to which this paper was a supplement.

Lemma 5. *If $\varphi \perp P$ for all P then φ is zero.*

We now set about proving that any automorphic representation π is a constituent of some $\operatorname{Ind} \sigma$. We may realize π on V/U, where U, V are subspaces of the space $\mathcal{A}$ of automorphic forms and V is generated by a single automorphic form φ. Totally order the conjugacy classes of parabolic subgroups in such a way that $\{P\} < \{P'\}$ implies rank $P \leq$ rank P' and rank $P <$ rank P' implies $\{P\} < \{P'\}$. Given φ let $\{P_\varphi\}$ be the minimum $\{P\}$ for which $\{P\} < \{P'\}$ implies $\varphi \perp P'$. Amongst all the φ serving to generate π choose one for which $\{P\} = \{P_\varphi\}$ is minimal. If $\psi \in \mathcal{A}$ let $\psi_P(g) = \psi_P(1, g)$ and consider the map $\psi \to \psi_P$ on V. If U and V had the same image we could realize π as a constituent of the kernel of the map. But this is incompatible with our choice of φ and hence if U_P and V_P are the images of U and V we can realize π in the quotient V_P/U_P.

Let $\mathcal{A}_P^\circ$ be the space of smooth functions ψ on $N(\mathbf{A})P(F)\backslash G(\mathbf{A})$ satisfying the following conditions.

(a) ψ is K-finite.

(b) For each g the function $m \to \psi(m, g) = \psi(mg)$ is automorphic and cuspidal.

Then $V_P \subseteq \mathcal{A}_P^\circ$. Since there is no point in dragging the subscript P about, we change notation, letting π be realized on V/U with $U \subseteq V \subseteq \mathcal{A}_P^\circ$. We suppose that V is generated by a single function φ.

Lemma 6. *Let A be the center of M. We may so choose φ and V that there is a character χ of $A(\mathbf{A})$ satisfying $\varphi(ag) = \chi(a)\varphi(g)$ for all $g \in G(\mathbf{A})$ and all $a \in A(\mathbf{A})$.*

Since $P(\mathbf{A})\backslash G(\mathbf{A})/K$ is finite, Lemma 3 implies that any φ in $\mathcal{A}_P^\circ$ is $A(\mathbf{A})$-finite. Choose V and the φ generating it to be such that the dimension of the span Y of $\{l(a)\varphi | a \in A(\mathbf{A})\}$ is minimal. Here $l(a)$ is left translation by a. If this dimension is one the lemma is valid. Otherwise there is an $a \in A(\mathbf{A})$ and $\alpha \in \mathbf{C}$ such that $0 < \dim(l(a) - \alpha)Y < \dim Y$. There are two possibilities. Either $(l(a) - \alpha)U = (l(a) - \alpha)V$ or $(l(a) - \alpha)U \neq (l(a) - \alpha)V$. In the second case we may replace φ by $(l(a) - \alpha)\varphi$, contradicting our choice. In the first we can realize π as a subquotient of the kernel of $l(a) - \alpha$ in V.

What we do then is to choose a lattice B in $A(\mathbf{A})$ such that $BA(F)$ is closed and $BA(F)\backslash A(\mathbf{A})$ is compact. Amongst all those φ and V for which Y has the minimal possible dimension we choose one φ for which the subgroup of B defined as $\{b \in B | l(b)\varphi = \beta\varphi, \beta \in \mathbf{C}\}$ has maximal rank. What we conclude from the previous paragraph is that this rank must be that of B. Since φ is invariant under

$A(F)$ and $BA(F)\backslash A(\mathbf{A})$ is compact, we conclude that Y must now have dimension one. The lemma follows.

Choosing such a φ and V we let ν be that positive character of $M(\mathbf{A})$ which satisfies

$$\nu(a) = |\chi(a)|, \quad a \in A(\mathbf{A}),$$

and introduce the Hilbert space $L_2^\circ = L_2^\circ(M(F)\backslash M(\mathbf{A}), \chi)$ of all measurable functions ψ on $M(\mathbf{Q})\backslash M(\mathbf{A})$ satisfying the following conditions.

(i) For all $m \in M(\mathbf{A})$ and all $a \in A(\mathbf{A}), \psi(ag) = \chi(a)\psi(g)$.

(ii) $\int_{A(\mathbf{A})M(\mathbf{Q})\backslash M(\mathbf{A})} \nu^{-2}(m)|\psi(m)|^2 dm < \infty$.

L_2° is a direct sum of irreducible invariant subspaces and if $\psi \in V$ then $m \to \psi(m, g)$ lies in L_2° for all $g \in G(\mathbf{A})$. Choose some irreducible component H of L_2° on which the projection of $\psi(\cdot, g)$ is not zero for some $g \in G(\mathbf{A})$.

For each ψ in V define $\psi'(\cdot, g)$ to be the projection of $\psi(\cdot, g)$ on H. It is easily seen that, for all $m_1 \in M(\mathbf{A}), \psi'(mm_1, g) = \psi'(m, m_1 g)$. Thus we may define $\psi'(g)$ by $\psi'(g) = \psi'(1, g)$. If $V' = \{\psi' | \psi \in V\}$, then we realize π as a quotient of V'. However if δ^2 is the modular function for $M(\mathbf{A})$ on $N(\mathbf{A})$ and σ the representation of $M(\mathbf{A})$ on H then V' is contained in the space of $\mathrm{Ind}\ \delta^{-1}\sigma$.

To prove the converse, and thereby complete the proof of the proposition, we exploit the analytic continuation of the Eisenstein series associated to cusp forms. Suppose π is a representation of the global Hecke algebra $\mathcal{H}$, defined with respect to some maximal compact subgroup K of $G(\mathbf{A})$. Choose an irreducible representation k of K which is contained in π. If E_κ is the idempotent defined by κ let $\mathcal{H}_\kappa = E_\kappa \mathcal{H} E_k$ and let π_κ be the irreducible representation of $\mathcal{H}_\kappa$ on the κ-isotypical subspace of π. To show that π is an automorphic representation, it is sufficient to show that π_κ is a constituent of the representation of $\mathcal{H}_\kappa$ on the space of automorphic forms of type κ. To lighten the burden on the notation, we henceforth denote π_κ by π and $\mathcal{H}_\kappa$ by $\mathcal{H}$.

Suppose P and the cuspidal representation σ of $M(\mathbf{A})$ are given. Let L be the lattice of rational characters of M defined over F and let $L_\mathbf{C} = L \otimes \mathbf{C}$. Each element μ of $L_\mathbf{C}$ defines a character ξ_μ of $M(\mathbf{A})$. Let I_μ be the κ-isotypical subspace of $\mathrm{Ind}\ \xi_\mu\sigma$ and let $I = I_0$. We want to show that if π is a constituent of the representation of I then π is a constituent of the representation of $\mathcal{H}$ on the space of automorphic forms of type κ.

If $\{g_i\}$ is a set of coset representatives for $P(\mathbf{A})\backslash G(\mathbf{A})/K$ then we may identify I_μ with I by means of the map $\varphi \to \varphi_\mu$ with

$$\varphi_\mu(nmg_ik) = \xi_\mu(m)\varphi(nmg_ik).$$

In other words we have a trivialization of the bundle $\{I_\mu\}$ over $L_{\mathbf{C}}$, and we may speak of a holomorphic section or of a section vanishing at $\mu = 0$ to a certain order. These notions do not depend on the choice of the g_i, although the trivialization does.

There is a neighborhood U of $\mu = 0$ and a finite set of hyperplanes passing through U such that for μ in the complement of these hyperplanes in U the Eisenstein series $E(\varphi)$ is defined for φ in I_μ. To make things simpler we may even multiply E by a product of linear functions and assume that it is defined on all of U. Since it is only the modified function that we shall use, we may denote it by E, although it is no longer the true Eisenstein series. It takes values in the space of automorphic forms and thus $E(\varphi)$ is a function $g \to E(g, \varphi)$ on $G(\mathbf{A})$. It satisfies

$$E(\rho_\mu(h)\varphi) = r(h)E(\varphi)$$

if $h \in \mathcal{H}$ and ρ_μ is $\operatorname{Ind} \xi_\mu\sigma$. In addition, if φ_μ is a holomorphic section of $\{I_\mu\}$ in a neighborhood of 0 then $E(g, \varphi_\mu)$ is a holomorphic in μ for each g, and the derivatives of $E(\varphi_\mu)$, taken pointwise, continue to be in $\mathcal{A}$.

Let I_r be the space of germs of degree r at $\mu = 0$ of holomorphic sections of I. Then $\varphi_\mu \to \rho_\mu(h)\varphi_\mu$ defines an action of $\mathcal{H}$ on I_r. If $s \leq r$ the natural map $I_r \to I_s$ is an $\mathcal{H}$-homomorphism. Denote its kernel by I_r^s. Certainly $I_0 = I$. Choosing a basis for the linear forms on $L_{\mathbf{C}}$ we may consider power series with values in the κ-isotypical subspace of $\mathcal{A}$, $\Sigma_{|\alpha|\leq r}\mu^\alpha\psi_\alpha$. $\mathcal{H}$ acts by right translation on this space and the representation so obtained is, of course, a direct sum of that on the κ-isotypical automorphic forms. Moreover $\varphi_\mu \to E(\varphi_\mu)$ defines an $\mathcal{H}$-homomorphism λ from I_r to this space. To complete the proof of the proposition all one needs is the Jordan-Hölder theorem and the following lemma.

Lemma 7. *For r sufficiently large the kernel of λ is contained in I_r^0.*

Since we are dealing with Eisenstein series associated to a fixed P and σ we may replace E by the sum of its constant terms for the parabolic associated to P, modifying λ accordingly. All of these constant terms vanish identically if and only if E itself does. If $Q_1, \ldots, Q_m$ is a set of representatives for the classes of parabolics associated to P let $E_i(\varphi)$ be the constant term along Q_i. We may suppose that M is a Levi factor of each Q_i. Define $\nu(m)$ for $m \in M(\mathbf{A})$ by $\xi_\mu(m) = e^{\langle \mu, \nu(m) \rangle}$. Thus $\nu(m)$ lies in the dual of L_R. If $\varphi \in I_\mu$, the function of $E_i(\varphi)$ has the form

$$E_i(nmg_jk, \varphi) = \sum_{\alpha=1}^{a} \sum_{\beta=1}^{b} p_\alpha(\nu(m)) \xi_{\nu_\beta(\mu)}(m) \psi_{\alpha\beta}(m,k).$$

Here $\psi_{\alpha\beta}$ lies in a finite-dimensional space independent of μ and g_j; ν_β, $1 \le \beta \le b$ is a holomorphic function of μ; and $\{p_\alpha\}$ is a basis for the polynomials of some degree. This representation may not be unique. The next lemma implies that we may shrink the open set U and then find a finite set $h_1, \ldots, h_n$ in $G(\mathbf{A})$ such that $E(\varphi_\mu)$ is 0 for $\mu \in U$, $\varphi_\mu \in I_\mu$ if and only if the numbers $E_i(h_j, \varphi_\mu), 1 \le i \le m$, $1 \le j \le n$ are all 0.

Lemma 8. *Let U be a neighborhood of 0 in $\mathbf{C}^l, \nu_1, \ldots, \nu_k$ holomorphic functions on U and $p_1, \ldots, p_a$ a basis for the polynomials of some given degree. Then there is a neighborhood V of 0 contained in U and a finite set $y_1, \ldots, y_b$ in $\mathbf{C}^l$ such that if $\mu \in V$ then*

$$(*) \qquad \Sigma p_i(y) e^{\nu j(\mu) \cdot y} = 0$$

for all y if and only if it is 0 for $y = y_1, \ldots, y_b$.

To prove this lemma one has only to observe that the analytic subset of U defined by the equation $(*)$, $y \in \mathbf{C}^l$, is defined in a neighborhood of 0 by a finite number of them.

We may therefore regard E as a function on U with values in the space of linear transformations from the space I, which is finite-dimensional, to the space C^{mn}. One knows from the theory of Eisenstein series that E_μ is injective for μ in an open subset of U. Then to complete the proof of the proposition, we need only verify the following lemma.

Lemma 9. *Suppose E is a holomorphic function in U, a neighborhood of 0 on $\mathbf{C}^l$, with values in $\mathrm{Hom}(I, J)$, where I and J are finite-dimensional spaces, and suppose that E_μ is injective on an open subset of U. Then there is an integer r such that if φ_μ is analytic near $\mu = 0$ and the Taylor series of $E_\mu \varphi_\mu$ vanishes to order r then $\varphi_0 = 0$.*

Projecting to a quotient of J, we may assume that $\dim I = \dim J$ and even that $I = J$. Let the first nonzero term of the power-series expansion of $\det E_\mu$ have degree s. Then we will show that r can be taken to be $s + 1$. It is enough to verify this for $l = 1$, for we can restrict to a line on which the leading term of E_μ still has degree s. But then multiplying E fore and aft by nonsingular matrices we may suppose it is diagonal with entries z^α, $0 \le \alpha \le s$. Then the assertion is obvious.

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References

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