L-Functions and Automorphic Representations*

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Introduction. There are at least three different problems with which one is confronted in the study of L-functions: the analytic continuation and functional equation; the location of the zeroes; and in some cases, the determination of the values at special points. The first may be the easiest. It is certainly the only one with which I have been closely involved.

There are two kinds of L-functions, and they will be described below: motivic L-functions which generalize the Artin L-functions and are defined purely arithmetically, and automorphic L-functions, defined by data which are largely transcendental. Within the automorphic L-functions a special class can be singled out, the class of standard L-functions, which generalize the Hecke L-functions and for which the analytic continuation and functional equation can be proved directly.

For the other L-functions the analytic continuation is not so easily effected. However all evidence indicates that there are fewer L-functions than the definitions suggest, and that every L-function, motivic or automorphic, is equal to a standard L-function. Such equalities are often deep, and are called reciprocity laws, for historical reasons. Once a reciprocity law can be proved for an L-function, analytic continuation follows, and so, for those who believe in the validity of the reciprocity laws, they and not analytic continuation are the focus of attention, but very few such laws have been established.

The automorphic L-functions are defined representation-theoretically, and it should be no surprise that harmonic analysis can be applied to some effect in the study of reciprocity laws. One recent small success was the proof of a reciprocity law for the Artin L-functions associated to tetrahedral representations of a Galois group and to a few other representations of degree two. It is the excuse for this lecture, but I do not want to overwhelm you with technique from harmonic analysis. Those who care to will be able to learn it at leisure from [6], in which a concerted effort was made to provide an introduction to automorphic representations, and so I forego proofs, preferring instead to review the evolution of our notion of an L-function and of a reciprocity law over the past five decades.

Artin and Hecke L-functions. An L-function is, first of all, a function defined by a Dirichlet series with an Euler product, and is therefore initially defined in a right half-plane. I will forbear defining explicitly the best known L-functions, the zeta-functions of Riemann and Dedekind, and the L-functions of Dirichlet, and begin with the more general functions introduced in this century by Hecke [19] and by Artin [2]. Artin's reciprocity law is the pattern to which all others, born and unborn, are cut.

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Although they overlap, the two kinds of L-functions are altogether different in their origins. If F is a number field or, if one likes, a function field, although I prefer to leave function fields in the background, for they will be discussed by Drinfeld [12], then a Hecke L-function is an Euler product $L(s,\chi)$ attached to a character χ of $F^\times \setminus I_F$. I_F is the group of idèles of F. If v is a place of F then $F_v^\times$ imbeds in I_F and χ defines a character χ_v of $F_v^\times$. To form the function $L(s,\chi)$ we take a product over all places of F:

$$L(s,\chi) = \prod_v L(s,\chi_v).$$

If v archimedean the local factor $L(s,\chi_v)$ is formed from Γ-functions. Here the important point is that whenever v is defined by a prime $\mathfrak{p}$ and χ_v is trivial on the units, as it is for almost all v, then

$$L(s,\chi_v) = \frac{1}{1-\alpha(\mathfrak{p})/N\mathfrak{p}^S}$$

with

$$\alpha(\mathfrak{p}) = \chi_v(\varpi_\mathfrak{p}),$$

$\varpi_\mathfrak{p}$ being a uniformizing parameter at $\mathfrak{p}$. The function $L(s,\chi)$ can be analytically continued and has a functional equation of the form

$$L(s,\chi) = \varepsilon(s,\chi)L(1-s,\chi)^{-1},$$

$\varepsilon(s,\chi)$ being an elementary function [35].

An Artin L-function is associated to a finite-dimensional representation ϱ of a Galois group $\mathrm{Gal}(K/F)$, K being an extension of finite degree. It is defined arithmetically and its analytic properties are extremely difficult to establish. Once again

$$L(s,\varrho) = \prod L(s,\varrho_v),$$

ϱ_v being the restriction of ϱ to the decomposition group. For our purposes it is enough to define the local factor when v is defined by a prime $\mathfrak{p}$ and $\mathfrak{p}$ is unramified in K. Then the Frobenius conjugacy class $\Phi_\mathfrak{p}$ in $\mathrm{Gal}(K/F)$ is defined, and

$$L(s,\varrho_v) = \frac{1}{\det(I-\varrho(\Phi_\mathfrak{p})/N\mathfrak{p}^s)} = \prod_{i=1}^{d} \frac{1}{1-\beta_i(\mathfrak{p})/N\mathfrak{p}^s},$$

if $\beta_1(\mathfrak{p}),\ldots,\beta_d(\mathfrak{p})$ are the eigenvalues of $\varrho(\Phi_\mathfrak{p})$.

Although the function $L(s,\varrho)$ attached to ϱ is known to be meromorphic in the whole plane, Artin's conjecture that it is entire when ϱ is irreducible and nontrivial is still outstanding. Artin himself showed this for one dimensional ϱ [3], and it can now be proved that the conjecture is valid for tetrahedral ϱ, as well as a few octahedral ϱ. Artin's method is to show that in spite of the differences in the definitions the function $L(s,\varrho)$ attached to a one-dimensional ϱ is equal to a Hecke L-function $L(s,\chi)$ where $\chi = \chi(\varrho)$ is a character of $F^\times \setminus I_F$. He employed all the available resources of class field theory, and went beyond them, for the equality of $L(s,\varrho)$

and $L(s,\chi(\varrho))$ for all ϱ is pretty much tantamount to the Artin reciprocity law, which asserts the existence of a homomorphism for I_F onto the Galois group Gal(K/F) of an abelian extension which is trivial on $F^\times$ and takes $\varpi_\mathfrak{p}$ to $\Phi_\mathfrak{p}$ for almost all $\mathfrak{p}$.

The equality of $L(s,\varrho)$ and $L(s,\chi)$ implies that of $\chi(\varpi_\mathfrak{p})$ and $\varrho(\Phi_\mathfrak{p})$ for almost all $\mathfrak{p}$. On close examination both these quantities are seen to be defined by elementary, albeit extremely complicated, operations, and so the reciprocity laws for one-dimensional ϱ, like the quadratic and higher reciprocity laws implicit in them, are ultimately elementary, and can for any ϱ and any given prime $\mathfrak{p}$ be verified by computation. The reciprocity law for tetrahedral ϱ seems, on the other hand, to be of a truly transcendental nature, and must be judged not by traditional criteria but by its success with the Artin conjecture.

Motivic L-functions. If V is a nonsingular projective variety over a number field then, for almost all $\mathfrak{p}$, V has a good reduction over the residue field $F_\mathfrak{p}$ at $\mathfrak{p}$ and we can speak of the number $N(n)$ of points with coordinates in the extension of F of degree n. Following Weil [36], we define the zeta-function $Z_\mathfrak{p}(s,V)$ by

$$\log Z_\mathfrak{p}(s,V)=\sum_{n=1}^{\infty}\frac{1}{n}\frac{N(n)}{N\mathfrak{p}^{ns}}.$$

We owe to the efforts of Dwork, Grothendieck, Deligne, and others the proof that

$$Z_\mathfrak{p}(s,V)=\prod_{i=0}^{2d}\prod_{j=1}^{b_i}\left(1-\frac{\alpha_{ij}(\mathfrak{p})}{N\mathfrak{p}_S}\right)^{(-1)^{i+1}}=\prod_{i=0}^{2d}L^i_\mathfrak{p}(s,V)^{(-1)^i}.$$

Here d is the dimension of V, b_i its i^{th} Betti number, and

$$|\alpha_{ij}(\mathfrak{p})|=N\mathfrak{p}^{i/2}.$$

It seems to have been Hasse (cf. [18]) who first proposed, in the case of an elliptic curve, the problem of proving that the L-function $L^i(s,V)$ defined by the Euler product

$$\prod_v L^i_v(s,V)$$

has analytic continuation and functional equation. Of course a solution of the problem involves a reasonable definition of the local factors at the infinite places and at the finite places at which V does not have good reduction.

Since b^i is generally greater than 1 and $L^i(s,V)$ is an Euler product of degree b^i, it cannot, except in special circumstances, be equal to an $L(s,\chi)$. Sometimes, however, $L^i(s,V)$ can be factored into a product of b^i Euler products of degree 1, each of which is equal to a Hecke L-function. The idea of factoring an L-function into Euler products of smaller degree is very important. It led Artin from the zeta-function of K to the L-functions associated to representations of Gal(K/F). Allusions to the same idea can be found in the correspondence of Dedekind with Frobenius [9], from which it appears that it was at the origin of the notion of a group character. The factorization can be simply interpreted in the context of the l-adic representations of Grothendieck.

The field K is the function field of an algebraic variety of dimension 0 over F and the zeta-function of K is $L^0(s,V)$. The variety V_F obtained from V by extension of scalars to the algebraic closure $\bar{F}$ has $[K:F]$ points. The Galois group Gal$(\bar{F}/F)$ acts on these points and hence on the l-adic étale cohomology group $H^0(V_F)$. The zeta-function may be defined in the same way as the Artin L-function except that it is now associated to the representation of Gal$(\bar{F}/F)$ on $H^0(V_F)$. The function field of V_F is $K \otimes_F \bar{F}$ and the action of Gal(K/F) we associate an operator $T(\sigma)$ on $H^0(V_F)$. If some linear combination of the $T(\sigma)$ is an idempotent E, we can restrict the representation of Gal$(\bar{F}/F)$ to its range, and employing Artin's procedure attach an L-function $L(s,E)$ to the restriction. Taking a family of such idempotents, orthogonal and summing to the identity, we obtain a factorization of $L^0(s,V)$ or of the zeta-function of K. Since the representation of Gal(K/F) and Gal$(\bar{F}/F)$ is that between the left and right regular representations, we obtain the factorization of Artin

$$L^0(s,V) = \zeta_K(s) = \prod_{\varrho} L(s,\varrho)^{\deg \varrho}.$$

The product is taken over all irreducible representations of Gal(K/F).

For a general variety the function $L^i(s,V)$ is obtained from the representation of Gal$(\bar{F}/F)$ on the l-adic cohomology group $H^i(V_F)$. The algebraic correspondences of V with itself which are of degree 0 and defined over F will define operators on $H^i(V_F)$ which commute with Gal$(\bar{F}\setminus F)$. Once again, if some linear combination of these operators is an idempotent E we may introduce $L(s,E)$, hoping that it will have an analytic continuation, and that it will be equal to Hecke L-function if the range of E has dimension one.

In particular, if we can write the identity as a sum of such idempotents which are orthogonal and of rank one then we can hope to prove that $L^i(s,V)$ is a product of Hecke L-functions, and so has the analytic continuation and a functional equation. The major examples here are abelian varieties of CM-type, the relevant endomorphisms being defined over F. The idempotents are constructed from these endomorphisms. The theorems were proved by Shimura, Taniyama, Weil, and Deuring (cf. [33]).

The functions $L(s,E)$ seem to be the correct, perhaps the ultimate, generalizations of the Artin L-functions. There is no reason to expect that they can be further factored. On close examination, it will be seen that the meaning of E has been left fuzzy. It should be a motive, a problematical notion, which Grothendieck has made precise ([23], [29]). But it cannot be shown to have all the properties desired of it without invoking certain conjectures closely related to the Hodge conjecture. Indeed, if the Hodge conjecture itself turns out to be false the notion will lose much of its geometric appeal. Furthermore there are L-functions arising in the study of Shimura varieties which we would be unwilling to jettison but which have not been shown to be carried by a motive in the sense of Groethendieck. But the notion is indispensable, and if the attendant problems will not yield to a vigorous assault then we have to prepare for a long siege.

If the functions $L(s,E)$ cannot be factored further than the theorems of Artin and Shimura-Taniyama mark the limits of usefulness of the Hecke L-functions in the study of the motivic L-functions. Fortunately the Hecke L-functions can be generalized.

Standard L-functions and the principle of reciprocity. If A is the adéle ring of F then I_F is $GL(1,\mathbf{A})$, $F^\times$ is $GL(1,F)$, and a character of $F^\times \setminus I_F$ is nothing but a representation of $GL(1,\mathbf{A})$ that occurs in the space of continuous functions on $GL(1,F)\setminus GL(1,\mathbf{A})$. It is the simplest type of autmorphic representation. $GL(n,\mathbf{A})$ acts on the factor space $GL(n,F)\setminus GL(n,\mathbf{A})$ and hence on the space of continuous functions on it. An automorphic representation of $GL(n,\mathbf{A})$ is basically an irreducible constituent π of the representation on the space of continuous functions, but the topological group $GL(n,\mathbf{A})$ is not compact and π is, in general, infinite-dimensional. So some care must be taken with the definitions [7]. One can attach to an automorphic representation π of $GL(n,\mathbf{A})$ an L-function $L(s,\pi)$ that will have an analytic continuation and a functional equation [17]:

$$L(s,\pi)=\varepsilon(s,\pi)L(1-s,\tilde{\pi}),$$

with $\tilde{\pi}$ contragredient to π. It is possible [14] to write π as a tensor product $\pi=\otimes_v\pi_v$, the product being taken over all places of F, and $L(s,\pi)$ is an Euler product $\prod_v L(s,\pi_v)$. At a finite place $v=\mathfrak{p}$

$$L(s,\pi_v)=\prod_{i=1}^{n}\frac{1}{1-\alpha_i(\mathfrak{p})\setminus N\mathfrak{p}^s}$$

is of degree n, and for almost all $\mathfrak{p}$ the matrix

$$A(\pi_v)=\begin{pmatrix}\alpha_1(\mathfrak{p}) & & 0\\ & \ddots & \\ 0 & & \alpha_n(\mathfrak{p})\end{pmatrix}$$

is invertible.

Since these L-functions, called standard, come in all degrees, there is no patently insurmountable obstacle to showing that each $L(s,E)$ is equal to some standard L-functions, thereby demonstrating the analytic continuation of $L(s,E)$. But the difficulties to overcome before this general principle of reciprocity is established are enormous, new ideas are called for, and little has yet been done.

If $F=\mathbf{Q}$, an automorphic representation of $GL(2,\mathbf{A})$ is an ordinary automorphic form, analytic or nonanalytic, in disguise, and the L-functions $L(s,\pi)$ have been with us for almost half a century. They were introduced and studied by Hecke [20], and later defined for nonanalytic forms by Maaß [28]. Moving from $n=1$ to $n=2$ does not give us much more latitude, but there are two obvious kinds of motivic L-functions of degree two.

If V is an elliptic curve then $L^1(s,V)$ is of degree two and the possibility that it would be equal to a standard L-function was first raised by Taniyama and later by Weil [37], during his re-examination of Hecke's theory. The numerical evidence is good, but no theoretical progress has been made with the problem, except over function fields where it is solved [10].

If ϱ is a two-dimensional representation of $\mathrm{Gal}(K/F)$ then the Artin L-function $L(s,\varrho)$ is of degree two. If ϱ is reducible or dihedral, Artin's theorem can deal with $L(s,\varrho)$. Otherwise the image of $\mathrm{Gal}(K/F)$ in $PGL(2,\mathbf{C}) = SO(3,\mathbf{C})$ is tetrahedral, octahedral, or icosahedral. One example of an icosahedral representation with a reciprocity law has been found **[8]**, but no general theorems are available. I shall return to the tetrahedral and octahedral below, after the principle of functoriality has been described.

The first successful applications of standard L-functions of degree two to the study of zeta-functions of algebraic varieties were for curves V obtained by dividing the upper half-plane by an arithmetic group, either a congruence subgroup of $SL(2,\mathbf{Z})$ or a group defined by an indefinite quaternion algebra ([13], [32]). Here $L^1(s,V)$ is a product of several $L(s,\pi)$ and the situation is similar to that for curves whose Jacobian is of CM-type, except that standard L-functions of degree two replace the Hecke L-functions, which are of degree one. The projections underlying the factorizations are linear combinations of the Hecke correspondences

It is not surprising that these varieties were handled first, for they are defined by a group, and the mechanism which links their zeta-functions with automorphic L-functions is relatively simple, similar to that appearing in the study of cyclotomic extensions of the rationals. There is a great deal to be learned from the study of these varieties and their generalizations, the Shimura varieties, but there are no Shimura varieties attached to $GL(n)$ when $n > 2$, and we must pass to more general groups.

Automorphic L-functions and the principle of functoriality. If G is any connected, reductive group over a global field an automorphic representation of $G(A)$ is defined as for $GL(n)$. The study of Eisenstein series led to a plethora of L-functions attached to automorphic representations. The Artin L-functions and the Hecke L-functions are fused in the class of automorphic L-functions, which contains them both, but the general automorphic L-function is in fact a kind of mongrel object, the true generalization of the Artin L-functions being the motivic L-functions and the true generalization of the Hecke L-functions being the standard L-functions.

To define the automorphic L-functions one associates to each connected, reductive group G over F an L-group ${}^LG = {}^LG_F$ ([5], [25]), itself an extension

$$1 \to {}^LG^0 \to {}^LG \to \mathrm{Gal}(K/F) \to 1$$

with ${}^LG^0$ a connected, reductive, complex group. K is simply a finite but large Galois extension of F. To each continuous finite-dimensional representation ϱ of ${}^LG$ which is complex-analytic on ${}^LG^0$ and each automorphic representation π of $G(A)$ one attaches an L-function $L(s,\pi,\varrho)$, which is an Euler product of degree equal to dimension of ϱ. There is evidence to support the hypothesis that each $L(s,\pi,\varrho)$ can be analytically continued to the whole plane as a meromorphic function with few poles and a functional equation.

The representation π is again a tensor product $\pi = \otimes_v \pi_v$ and

$$L(s,\pi,\varrho) = \prod_v L(s,\pi_v,\varrho)$$

For almost all finite v the theory of spherical functions, or, if one prefers, of Hecke operators, attaches to π_v a conjugacy class $\{g_v\} = \{g(\pi_v)\}$ in ${}^LG$ which reduces to the Frobenius class when $G = \{1\}$. The local factor for these places is

$$L(s,\pi,\varrho) = \frac{1}{\det(1-\varrho(g_v)/N\mathfrak{p}^s)}$$

if v is defined by $\mathfrak{p}$. If G is $GL(n)$ then ${}^LG$ is a direct product $GL(n,\mathbf{C}) \times\times$ Gal(K/F) and the projection of $\{g_v\} = \{g(\pi_v)\}$ on the first factor is the class of $A(\pi_v)$. Consequently if ϱ is the projection on the first factor then $L(s,\pi,\varrho)$ is the standard L-function $L(s,\pi)$.

The automorphic L-functions once defined, their resemblance to the Artin L-functions is manifest, and the possibility suggests itself of establishing their analytic continuation by showing that when G, π and ϱ are given there is a representation π' of $GL(n,A)$ with $n = \deg\varrho$ such that $\{A(\pi'_v)\} = \{\varrho(g(\pi_v))\}$ for almost all v and

$$L(s,\pi,\varrho) = L(s,\pi').$$

For $G = \{1\}$ this would be the reciprocity law for Artin L-functions.

More generally, if H and G are two connected reductive groups over F and we have a commutative diagram

$$\begin{tikzcd} {}^LG \arrow[d, "\varphi"] \arrow[dr] & \\ {}^LH \arrow[r] & \mathrm{Gal}(K\setminus F) \end{tikzcd}$$

with φ complex-analytic, then to every automorphic representation π of ${}^LG$ there should be an automorphic representation π' of H which is such that $\{g(\pi'_v)\} = \{\varphi(g(\pi_v))\}$ for almost all v. There is evidence that this is so, although some subtleties must be taken into account. I refer to the phenomenon as the principle of functoriality in the L-group.

Examples. Suppose E is finite extension of F. Then G is also a group over E and the L-group over E, ${}^LG_E$, is a subgroup of ${}^LG_F$. It is the inverse image of Gal(K/E) in ${}^LG_F$. The principle of functoriality implies the possibility of making a change of base from F to E and associating to each automorphic representation π of $G(A_F)$ an automorphic representation Π of $G(A_E)$, sometimes called a lifting of π. For almost all places, w, of E the class $\{g(\Pi_w)\}$ must be $\{g(\pi_v)^f\}$ if w divides the place v of F and $f = [E_w : F_v]$.

Ideas of Saito [30] and Shintani [34] allow us to show that base change is always possible when $G = GL(2)$ and E is a cyclic extension of prime degree are enough, and for them it is possible to characterize those Π which are liftings. The Galois group Gal(E/F) acts on A_E and on $GL(2,\mathbf{A}_E)$ and thus on the set of automorphic representations of $GL(2,A_E)$. Apart from some trivial exceptions, Π is a lifting if and only if Π is fixed by Gal(E/F).

Base change is a first step towards a proof of the principle of functoriality and Artin's conjecture for two-dimensional representations. Suppose, for example, that σ is a tetrahedral representation of Gal$(\bar{F}/F)$. Then there is a cyclic extension E of F of degree three which is such that the restriction Σ of σ to Gal$(\bar{F}/E)$ is dihedral. Consequently the principle of functoriality applies to it and yields an automorphic representation $\Pi = \Pi(\Sigma)$ of $GL(2, A_E)$. The class of Σ is invariant under Gal(E/F) and therefore Π is too, and is a lifting. There is precisely one representation π which lifts to Π and has central character $\det \sigma$. It should be $\pi(\sigma)$, the representation whose existence is demanded by the principle of functoriality. At first sight this does not look difficult to show, for the eigenvalues of $\sigma(\Phi_{\mathfrak{p}})$ and $\{A(\pi_v)\}$, where v is the place defined by $\mathfrak{p}$, differ only by cube roots of unity, but it should be a deeper matter. However fortune smiles on us, for we can deduce some interesting theorems without pressing for a full understanding.

There are two ways of proceeding. The one used in [26] has the disadvantage that it does not work for all fields or all tetrahedral representations, but the advantage that it also works for some octahedral representations. It invokes a theorem of Deligne-Serre, characterizing some of the automorphic representations attached to two-dimensional representations of the Galois group. The other (cf. [15]) employs special cases of the principle of functoriality proved by Piatetski-Shapiro and Gelbart-Jacquet.

One begins with Serre's observation to me that composition of σ with the adjoint representation φ of $GL(2)$ on the Lie algebra of $PGL(2)$ gives a three-dimensional monomial representation ϱ to which, by a theorem of Piatetski-Shapiro [21], the principle functoriality applies to yield an automorphic representation $\pi(\varrho)$ of $GL(3, \mathbf{A}_F)$. On the other hand, the L-group of $GL(2)$ is a direct product $GL(2, \mathbf{C})\times$ Gal(K/F) and that of $GL(3)$ is a direct product $GL(3, \mathbf{C})\times$ Gal(K/F). The principle of functoriality should attach to the homomorphism

$$\varphi \times \mathrm{id} : GL(2, \mathbf{C}) \times \mathrm{Gal}(K/F) \to GL(3, \mathbf{C}) \times \mathrm{Gal}(K/F)$$

a map φ_* from automorphic representations of $GL(2, \mathbf{A}_F)$ to automorphic representations of $GL(3, \mathbf{A}_F)$. The existence of φ_* has been proven by Gelbart-Jacquet [16].

If the principle of functoriality is consistent and π is $\pi(\sigma)$ then $\varphi_*(\pi)$ must be $\pi(\varrho)$. Conversely, elementary considerations, which exploit the absence of an element of order six in the tetrahedral group, show that if $\varphi_*(\pi)$ equals $\pi(\varrho)$ then π is $\pi(\sigma)$. That $\varphi_*(\pi)$ equals $\pi(\varrho)$ follows easily from an analytic criterion of Jacquet-Shalika [22].

Even for $GL(2)$ base change for cyclic extensions is not proved without some effort, the principal tools being the trace formula and the combinatorics of the Bruhat-Tits building. These are being developed by Arthur [1] and by Kottwitz [23], but our knowledge of harmonic analysis is still inadequate to a frontal attack on the problem of base change for a general group. Nonetheless some progress can be expected, although it is not clear how close base change will bring us to the Artin conjecture.

For number fields there has been no other recent progress with the principle of reciprocity. But we could also try to show that a motivic L-function is equal to an automorphic L-function $L(s, \pi, \varrho)$ which is not standard or to a product of such functions. This may not imply the analytic continuation of $L(s, E)$ but can have concrete arithmetic consequences and the proof may direct our attention to important features of the mechanism underlying the principles of reciprocity and functoriality [31].

The immediate examples are the L-functions defined by Shimura varieties [27]. These varieties are a rich source of ideas and problems, but once again we must advance slowly, deepening our understanding of harmonic analysis and arithmetic as we proceed. For the varieties associated to the group over **Q** obtained by restriction of scalars from a totally indefinite quaternion algebra over a totally real field F, the problems are tractable. In [27] no motives are mentioned, but the zeta-function is expressed as a quotient of products of automorphic L-function of degree 2^n, where $n = [F : Q]$ is the dimension of the variety. For $n = 2$, the analytic continuation and functional equation have been established by Asai [4], and we have the first examples of analytic continuation for motivic L-functions which are of degree four and, apparently, irreducible and not induced.

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