Orbital Integrals on Forms of SL(3), I

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Dedicated to André Weil on his 77th birthday.

If G is a reductive group over a local or global field then it is possible to attach to G the endoscopic groups studied by Shelstad ([16] and the references therein) and myself ([5]). Over the real and complex fields the introduction of these groups has been amply justified by the work of Shelstad, but over other fields their value remains doubtful as long as one is unable to establish the possibility of a transfer of orbital integrals ([5]).

If T is a Cartan subgroup of G over the local field F, supposed of characteristic zero, and γ in $T(F)$ is regular then the orbital integral of a function f, taken for simplicity to be smooth and of compact support, over the orbit of γ is

$$\Phi(\gamma, f) = \int_{T(F)\backslash G(F)} f(g^{-1}\gamma g)dg.$$

Let $\bar{F}$ be the algebraic closure of F and $\mathfrak{A}(T/F)$ the set of all $a \in G(\bar{F})$ such that $\epsilon_\sigma = \sigma(a)a^{-1} \in T(\bar{F})$ for all $\sigma \in \mathrm{Gal}(\bar{F}/F)$. Then $a \to \epsilon = \{\epsilon_\sigma\}$ yields an imbedding of $\mathcal{D}(T/F) = T(\bar{F})\backslash\mathfrak{A}(T/F)/G(F)$ in the image $\mathcal{E}(T/F)$ of $H^1(F, T_{\mathrm{sc}})$ in $H^1(F, T)$, the group T_{sc} being the inverse image of T in the simply-connected covering of the derived group of G. If ϵ in $\mathcal{E}(T/F)$ is the image of an a in $\mathfrak{A}(T/F)$ we set

$$\Phi(\gamma, \epsilon, f) = \Phi(a^{-1}\gamma a, f),$$

but if ϵ does not lie in the image of $\mathfrak{a}(T/F)$ we set

$$\Phi(\gamma, \epsilon f) = 0.$$

Finally if κ is a character of the group $\mathcal{E}(T/F)$ we set

$$\Phi^\kappa(\gamma, f) = \sum_\epsilon \kappa(\epsilon)\Phi(\gamma, \epsilon, f).$$

It appears that to transfer orbital integrals one needs methods for studying the asymptotic behavior of $\Phi^\kappa(\gamma, f)$ as γ approaches a singular element. Over the real and complex fields the differential equations it satisfies provide an effective tool ([2]). Over non-archimedean fields the germ theory of Shalika is available, but this is only a first step. However, as Shalika himself pointed out to me, there is a technique at hand for the study of the asymptotic behavior of integrals on non-archimedean manifolds, that of Igusa ([4]).

It will be described in detail in Section 1 and applies to integrals along fibres in a fibering over a curve, but the fibering, the form defining the measure, and the integrand must satisfy a number of conditions which, especially those imposed on the fibering, are not easy to create.

The purpose of this paper, which is frankly tentative, is to construct Igusa fiberings which are applicable to the study of orbital integrals on forms of $SL(3)$. The method is to modify a resolution due to Grothendieck-Springer ([1]) of the morphism from G to the variety of semi-simple orbits. The construction is carried out in Section 2, Section 3 and Section 4 using simple techniques of algebraic geometry. In Section 5 it is verified that the orbital integrals are defined by integrals of Igusa integrands over the fibres. In a second paper, to be written jointly with Shelstad, it will be shown, all being well, that for the small class of groups being considered the results allow the introduction of transfer factors satisfying the conditions of [5].

The present results are thus extremely modest, and can only be a token of my esteem for the mathematical achievement of André Weil, but it is a token gladly given.

1. Asymptotic behavior of integrals. In this paragraph we review some results of Igusa [4] and develop them in a form suitable for our purposes. We introduce first of all an Igusa fibering, which is a morphism φ from a smooth variety Y to a smooth, but not necessarily complete curve C on which there is a distinguished point s_0. The point, the two varieties, and the morphism φ are all to be defined over a given ground field F, and φ is to be smooth outside $\varphi^{-1}(s_0)$. In addition if $y_0 \in \varphi^{-1}(s_0)$ and λ is a local coordinate on Z at s_0 then there are local coordinates $\mu_1, \dots, \mu_n$ on Y at y_0 which are defined over F and such that φ is given by an equation

$$\lambda = \alpha \mu_1^{a_1} \cdots \mu_n^{a_n}$$

with $a_i \in \mathbf{Z}, a_i \geq 0$, and α regular and invertible at y_0. Such local coordinates will be called admissible. The inverse image $\varphi^{-1}(s_0)$ is the union of divisors which are smooth, apart from possible self-intersections, and cross normally. Let $\mathfrak{e}$ be the set of these divisors and let $a(E)$ be multiplicity of E in $\varphi^{-1}(s_0)$.

A form ω defined over $\bar{F}$, regular, and of maximal degree will be called an Igusa form if the divisor of its zeros is contained in $\varphi^{-1}(s_0)$. Thus to each divisor E in $\mathfrak{E}$ we can so associate an integer $b(E) > 0$ that the divisor of zeros is $\Pi_{E \in \mathfrak{e}} E^{b(E)-1}$. If $y_0 \in \varphi^{-1}(s_0)$ then in terms of the local coordinates $\mu_1, \dots, \mu_n$ the form can be written as

$$\omega = W(\mu_1, \dots, \mu_n) \mu_1^{b_1 - 1} \cdots \mu_n^{b_n - 1} d\mu_1 \cdots d\mu_n,$$

where $b_i = b(E)$ if $\mu_i = 0$ defines a branch of $E \in \mathfrak{E}$ at y_0 and $b_i = 1$ if $a_i = 0$.

If F is a local field then $Y(F)$ and $C(F)$ are F-manifolds. A smooth function f on $Y^0(F) = Y(F) - \varphi^{-1}(s_0)$ will be called an Igusa integrand if the following two conditions are satisfied:

(a) The closure of the support of f is proper over $C(F)$.

(b) To each $E \in \mathfrak{E}$ there is associated a character $\kappa(E)$ of $F^\times$ such that in a neighborhood of a point $y_0 \in \varphi^{-1}(s_0)$ the function f is defined by an equation

$$f(y) = \gamma\kappa_1(\mu_1)\cdots\kappa_n(\mu_n)$$

where $\kappa_1, \dots, \kappa_n$ are unitary characters of $F^\times$ with $\kappa_i = \kappa(E)$ if $\mu_i = 0$ defines a branch of E at y_0 but otherwise identically 1, and where γ is smooth at y_0. Thus if F is non-archimedean, which is the case with which we will be concerned in this paragraph, γ can be taken to be constant.

The fibering, the form, and the function taken together can be called Igusa data. If λ is a local coordinate on C at s_0 and if we choose a non-zero value λ_0 for it we can introduce coordinates $\mu_1, \dots, \mu_n$ at any point of $\varphi^{-1}(s(\lambda_0))$ such that $\mu_1 = \lambda$. Then $\mu_2, \dots, \mu_n$ serve as local coordinates on $\varphi^{-1}(s(\lambda_0)) = Y_{s(\lambda_0)}$ and the formula

$$\omega_{\lambda_0} = W(\lambda_0, \mu_2, \dots, \mu_n)d\mu_2\cdots d\mu_n = \frac{\omega}{\varphi^*(d\lambda)}$$

defines a form of maximal degree on $Y_{s(\lambda_0)}$.

We set

$$F(\lambda) = \int_{Y_{s(\lambda)}} f(y)|\omega_\lambda|.$$

The results of Igusa describe the asymptotic behavior of F as $\lambda \to 0$, and yield when they can be applied explicit formulas for the germs of Shalika. In order to state them we need some notation.

Let Θ be the collection of all pairs (θ, β), where θ is a unitary character of $F^\times$ and β a positive rational number, and let $\mathfrak{E}(\theta, \beta)$ be the set of all $E \in \mathfrak{E}$ such that $\theta^{a(E)} = \kappa(E)$ and $\beta = \beta(E) = \frac{b(E)}{a(E)}$. Let $e(\theta, \beta)$ be the maximal number of branches of divisors in $\mathfrak{E}(\theta, \beta)$ which cross at a point.

1.1. Proposition. *Let F be non-archimedean, q the number of elements in the residue field, and $m(\lambda) = -\log_q|\lambda|$. For $|\lambda|$ sufficiently small there is an expansion*

$$F(\lambda) = \Sigma\theta(\lambda)|\lambda|^{\beta-1}\sum_{r=1}^{e(\theta,\beta)} m(\lambda)^{r-1}F_r(\theta, \beta, f),$$

where the sum runs over all pairs (θ, β) for which $\mathfrak{E}(\theta, \beta)$ is not empty, and the constant $F_r(\theta, \beta, f)$ depends on $\theta, \beta,$ and f but not on λ.

We will indicate the proof of this, referring to [4] for details, even though the results are formulated differently there, but we want to explain at the same time how the constants $F_r(\theta, \beta, f)$ are calculated.

If $1 \le r \le e(\theta, \beta)$ let $D = D_r(\theta, \beta)$ be the variety which in the neighborhood of any point is obtained by taking the union of all intersections of r distinct branches of divisors in $\mathfrak{E}(\theta, \beta)$. The points of D through which pass exactly r branches of divisors in $\mathfrak{e}$ form an open, dense, and smooth subvariety $\widehat{D}$ of D. We now define a form $\nu = \nu_r = \nu_r(\theta, \beta)$ and a function $h = h_r = h_r(\theta, \beta)$ on $\widehat{D}$.

If $y \in D$ then we may so choose admissible local coordinates $\mu_1, \ldots, \mu_n$ at y that a given branch of D at y is given locally by $\mu_1 = \cdots = \mu_r = 0$. On this branch and near y the form is given by

$$\nu = \frac{W(0, \ldots, 0, \mu_{r+1}, \ldots, \mu_n)}{\alpha^\beta} \prod_{j=r+1}^{n} \mu_j^{b_j - \beta a_j - 1} d\mu_{r+1} \cdots d\mu_n$$

and the function by

$$h(0, \ldots, 0, \mu_{r+1}, \ldots, \mu_n) = \frac{\gamma}{\theta(\alpha)} \prod_{j=r+1}^{n} \kappa_j(\mu_j)\theta(\mu_j^{-a_j}).$$

Because β may not be integral the form ν may be multi-valued, but it is easily verified that the associated measure $|\nu|$ and the function h are well-defined on $\widehat{D}$.

1.2. Proposition. *The constant $F_r(\theta, \beta, f)$ is given by the principal value*

$$PV \int_D h|\nu|.$$

The principal value appearing here has still to be defined. To this purpose we fix the local coordinate λ and confine our attention to

$$Y^\epsilon = \{y \in Y(F) \mid \varphi(y) = s(\lambda), |\lambda| \le \epsilon\},$$

where ϵ is small. We agree that at a point $y_0 \notin \varphi^{-1}(s_0)$ all local coordinate systems are admissible. We cover the closure of the support of f by a finite number of disjoint coordinate patches U with admissible coordinates $\mu_1, \ldots, \mu_n$ satisfying the following conditions.

(i) There are integers $M_i, 1 \le i \le n$, such that U is given by the inequalities $|\mu_i| \le q^{-M_i}$. There is also an integer $s = s_U(\theta, \beta)$ such that $\mu_i = 0$ is a branch of a divisor in $\mathfrak{E}(\theta, \beta)$ if and only if $1 \le i \le s$.

(ii) If φ is given by $\lambda = \alpha\mu_1^{a_1} \cdots \mu_n^{a_n}$ then $|\alpha| = q^{-m}$ is constant on U.

(iii) If $\omega = W(\mu_1, \cdots, \mu_n)\mu_1^{b_1 - 1} \cdots \mu_n^{b_n - 1} d\mu_1 \cdots d\mu_n$ then $|W(\mu_1, \ldots, \mu_n)|$ is constant on U.

(iv) If $f = \gamma\kappa_1(\mu_1) \cdots \kappa_n(\mu_n)$ then γ is constant on U.

It is easy to see that any covering of the closure of the support of f can be refined to a covering satisfying these conditions and that any two such coverings have a common refinement. We shall define $PV \int_{D_r} h_r|\nu_r|$ as a sum

$$\sum_U PV \int_{D_r \cap U} h_r|\nu_r|.$$

If $D_r \cap U$ is empty the principal value is to be zero. So we suppose that $D_r \cap U$ is not empty and thus that $s \ge r$.

Let the principal part of

$$\prod_{i=1}^{s} \frac{1}{1 - t^{a_i}}$$

at $t = 1$ be

$$\sum_{j=1}^{s} \frac{c_j}{(1-t)^j}$$

and let $A(x)$ be the polynomial

$$\sum_{j=1}^{s} c_j (x+1)\cdots(x+j-1).$$

We define $A_r(y)$ by

$$A(x-y) = \sum_{r=1}^{s} x^{r-1} A_r(y)$$

and if

$$M = m + \sum_{i=1}^{s} a_i M_i + \sum_{i=s+1}^{n} a_i m(\mu_i)$$

we set

$$PV \int_{D_r \cap U} h_r |\nu_r| = \left(1 - \frac{1}{q}\right)^{s-1} PV \int_{D_s \cap U} A_r(M) h_s |\nu_s|.$$

The principal value on the right, in which the domain of integration is now $D_s \cap U$, remains to be defined.

1.3. Lemma. *If* $\mathrm{Re}\ z_j \gg 0, s < j \leq n,$ *then*

$$\int_{\widehat{D}_s \cap U} A_r(M) h_s \prod_{j=s+1}^{n} |\mu_j|^{z_j} |\nu_s| \tag{1.4}$$

is absolutely convergent, and defines a meromorphic function of $z_{s+1}, \ldots, z_n$ *which is analytic at* $z_{s+1} = \cdots = z_n = 0$.

Once the lemma is proven we can take

$$PV \int_{D_s \cap U} A_r(M) h_s |\nu_s|$$

to be the value at $z_{s+1} = \cdots = z_n = 0$ of the function defined by (1.4).

The integral is a sum of integrals of the form

$$\prod_{j=s+1}^{n} \int_{|\mu_j| \leq q^{-M_j}} m(\mu_j)^k \eta_j(\mu_j) |\mu_j|^{c_j + z_j - 1} |d\mu_j|,$$

where for each j either the unitary character η_j is not trivial or the real number c_j is different from 0. The integrals appearing in the product are clearly convergent for Re $c_j + z_j > 0$. If η_j is ramified the j^{th} integral is zero, but if η_j is unramified and ϵ_j its value at a generator of the maximal ideal it is

$$\left(1 - \frac{1}{q}\right) \left(\frac{-1}{\mathrm{In}\ q}\right)^k \frac{d^k}{dz_j^k} \left(\frac{\epsilon_j^{M_j} q^{-M_j(c_j+z_j)}}{1 - \frac{\epsilon_j}{q^{c_j+z_j}}}\right).$$

Thus the lemma is established.

We need to verify that the definition of principal value is independent of the covering chosen. The first step is to show that if we take a grid decomposition of a given patch U into patches V then

$$\sum_V PV \int_{D_r \cap V} h_r |\nu_r| = PV \int_{D_r \cap U} h_r |\nu_r|.$$

A grid decomposition is obtained by choosing for each i an integer $N_i \geq M_i$ and refining U by patches V with coordinates $\phi_i, \mu_i = \bar{\mu}_i + \phi_i, |\phi_i| \leq q^{-N_i}$, it being understood that $\bar{\mu}_i = 0$ if $|\bar{\mu}_i| \leq q^{-N_i}$. If some of the κ_i are ramified, then the condition (iv) will be satisfied only for $N_i - M_i$ sufficiently large, but that is a minor complication which we choose to ignore.

The refinement can be carried out in stages, taking $N_j = M_j + 1$ for one j and $N_i = M_i$ for $i \neq j$. We will have q patches $V_0, \cdots, V_{q-1}$ with $\bar{\mu}_j = \mu_j^k$ equal to 0 for $k = 0$ and unequal to 0 for $k \neq 0$. Suppose first that $j > s$. Then it has to be shown that

$$PV \int_{D_s \cap U} A_r(M) h_s |\nu_s| = \sum_k PV \int_{D_s \cap V_k} A_r(M) h_s |\nu_s|.$$

This follows readily from the definition once it is observed, as is implicit in the notation, that the functions $A_r(M)$ appearing on the two sides of the equality are indeed the same. On V_0 this is clear because there $\mu_j = \phi_j$. On $V_k, k > 0$,

$$\lambda = \alpha(\mu_j^k + \phi_j)^{a_j} \left(\prod_{i \neq j} \mu_i^{a_i} \right) \mu_j^{a'_j},$$

with $a'_j = 0$, and $\mu_i = \phi_i, i \neq j$. Thus on V_k the function α is replaced by

$$\alpha' = \alpha(\mu_j^k + \phi_j)^{a_j}$$

and

$$M' = m(\alpha') + \sum_{i=1}^{s} a_i M_i + \sum_{\substack{i=s+1 \\ i \neq j}}^{n} a_i m(\phi_i) + a'_j m(\phi_j) = M.$$

We next take $j \leq s$, supposing to simplify the notation that j is simply s. Then it must be shown that if B_r is defined like A_r but with respect to $a_1, \ldots, a_{s-1}$ rather than $a_1, \ldots, a_s$, if

$$M(\mu) = m + \sum_{i \neq s} a_i M_i + \sum_{i=s+1}^{n} a_i m(\mu_i) + a_s m(\mu),$$

and

$$N = m + \sum_{i=1}^{s} a_i N_i + \sum_{i=s+1}^{n} a_i m(\mu_i)$$

then

$$\left(1-\frac{1}{q}\right)A_r(M)=\left(1-\frac{1}{q}\right)A_r(N)+\int_{N_s>m(\mu)\geq M_s}B_r(M(\mu))\frac{|d\mu|}{|\mu|}.$$

Since $N=M+a_s$ and $M(\mu)=M$ on the domain of integration this reduces to

$$A_r(M)=A_r(M+a_s)+B_r(M), \tag{1.5}$$

where it is understood that $B_r\equiv 0$ if $r>s-1$.

The identity (1.5) is valid for all r if and only

$$\sum_{r=1}^{s}x^{r-1}A_r(y)-\sum_{r=1}^{s}x^{r-1}A_r(y+a_s)=\sum_{r=1}^{s-1}x^{r-1}B_r(y). \tag{1.6}$$

This identity need only be verified for y and x integral and x very large. The expression on the left is the coefficient of t^{x-y} in the Taylor expansion at $t=0$ of the principal part at $t=1$ of

$$\prod_{i=1}^{s}\frac{1}{1-t^{a_i}}-t^{a_s}\prod_{i=1}^{s}\frac{1}{1-t^{a_i}}=\prod_{i=1}^{s-1}\frac{1}{1-t^{a_i}},$$

which is by definition the right side of (1.6).

Suppose next that we have two coordinate systems $\phi_1,\cdots,\phi_n$ and $\mu_1,\cdots,\mu_n$ on the same patch U and that

$$\mu_i=f_i(\phi_1,\ldots,\phi_n)\phi_i,$$

if $\phi_i=0$ defines a branch of a divisor in $\mathfrak{E}(\beta,\theta)$, where $|f_i|=q^{-N_i}$ is constant on U.

We want to show that the two coordinate systems lead to the same principal value. It follows readily from the definition of

$$PV\int_{D_s\cap U}A_r(M)h_s|\nu_s|$$

that this will be so if the function M defined on $\widehat{D}_s\cap U$ by the coordinates $\mu_1,\cdots,\mu_n$ and the function M' defined by the coordinates $\phi_1,\ldots,\phi_n$ are equal. If near a point of $\widehat{D}_s$ we have

$$\lambda=\beta\prod_{i=1}^{s}\mu_i^{a_i}$$

then

$$M=m(\beta)+\sum_{i=1}^{s}a_iM_i$$

and if

$$\lambda=\beta'\prod_{i=1}^{s}\phi_i^{a_i}$$

then

$$M' = m(\beta') + \sum_{i=1}^{s} a_i M_i'.$$

However $M_i = M_i' + N_i$ and

$$\beta' = \beta \sum_{i=1}^{s} f_i^{a_i},$$

so that

$$m(\beta') = m(\beta) + \sum_{i=1}^{s} a_i N_i.$$

To show that the principal value is well defined and independent of the covering by coordinate patches, we observe that if we have two we can find a grid decomposition of one, that is of each of its patches, which is also a grid decomposition of the other. So we are immediately reduced to the case of a single patch U with two coordinate systems $\phi_1, \ldots, \phi_n, \mu_1, \ldots, \mu_n$. Renumbering the coordinates we may suppose that $\phi_i = 0$ and $\mu_i = 0$ define the same divisor for $1 \leq i \leq s_U(\beta, \theta)$. Thus $\mu_i = f_i(\phi_1, \ldots, \phi_n)\phi_i$ when $1 \leq i \leq s_U(\beta, \theta)$ and $|f_i| \neq 0$ on U. After passage to a finer grid decomposition the absolute values $|f_i|$ become constant on the patches. The independence is thus verified.

We now prove Propositions 1.1 and 1.2. It is enough to verify them for a function supported on a coordinate patch U on which it has the form

$$f(y) = \prod_{i=1}^{n} \kappa_i(\mu_i),$$

while

$$\omega = \prod_{i=1}^{n} \mu_i^{b_i - 1} d\mu_1 \cdots d\mu_n$$

and

$$\lambda = \alpha \mu_1^{a_1} \cdots \mu_n^{a_n}$$

with $|\alpha| = q^{-m}$ constant on U.

What has to be shown is that for each unitary character θ of $F^\times$ the function

$$\Theta(s) = \int_{|\lambda| \leq \epsilon} F(\lambda)\theta^{-1}(\lambda)|\lambda|^s |d\lambda|$$

is a rational function of $t = q^{-s}$ whose principal part at $t = q^\beta$ has a Taylor expansion at $t = 0$ in which the coefficient of t^n is

$$\left(1 - \frac{1}{q}\right) q^{-n\beta} \sum_{r=1}^{e(\theta,\beta)} n^{r-1} F_r(\theta, \beta, f)$$

for n large.

Following Igusa we obtain

$$\Theta(s) = \frac{|\alpha|^s}{\theta(\alpha)} \int_U f(y) \prod_{i=1}^n \theta(\mu_i^{-a_i}) |\mu_i|^{a_i s + b_i - 1} |d\mu_1| \cdots |d\mu_n|$$
$$= \frac{|\alpha|^s}{\theta(\alpha)} \prod_{i=1}^n \int_{|\mu| \le q^{-M_i}} \kappa_i(\mu) \theta^{-a_i}(\mu) |\mu|^{a_i s + b_i - 1} |d\mu|.$$

The integral will be zero if any of the characters $\kappa_i \theta^{-a_i}$ is ramified. Otherwise let ϵ_i be the value of $\kappa_i \theta^{-a_i}$ at a generator of the maximal ideal. Then

$$\Theta(s) = \frac{|\alpha|^s}{\theta(\alpha)} \left(1 - \frac{1}{q}\right)^n \prod_{i=1}^n \frac{(\epsilon_i q^{-a_i s - b_i})^{M_i}}{1 - \frac{\epsilon_i}{q^{a_i s + b_i}}}.$$

We suppose that for $1 \le i \le p$ we have $\kappa_i = \theta^{a_i}$ and $\frac{b_i}{a_i} = \beta$ but that for $i > \rho$ one of these two equalities fails to obtain. Then $\Theta(s)$ is the value at $z = 0$ of

$$\left(1 - \frac{1}{q}\right)^p \int_{|\mu_j| \le q^{-M_j}} \left\{ \prod_{i=1}^p \frac{(q^{-a_i(s+\beta)})^{M_i}}{1 - \frac{1}{q^{a_i(s+\beta)}}} \right\} \cdot |\alpha|^{s+\beta} h_p \prod_{j=p+1}^n |\mu_j|^{z + a_j s + b_j - 1} |d\mu_{p+1}| \cdots |d\mu_n|.$$

The procedure of forming the principal part at $t = q^\beta$ and then the coefficients of its Taylor series at $t = 0$ is linear, depends analytically on any parameter on which the function depends analytically, and is interchangeable with a passage to the limit. So we can carry it out under the integral sign.

If the principal part of

$$\prod_{i=1}^p \frac{1}{1 - t^{a_i}}$$

at $t = 1$ is

$$\sum_{j=1}^p \frac{c_j}{(1-t)^j}$$

and

$$A(x) = \sum_{j=1}^p c_j (x+1) \cdots (x+j-1)$$

then the Taylor expansion at $t = 0$ of the principal part of the function under the integral sign at $t = q^\beta$ is

$$\sum_{n=0}^\infty A(n) t^{n+M} q^{-(n+M)\beta} h_p \prod_{j=p+1}^n |\mu_j|^{z + b_j - \beta a_j - 1}$$

with

$$M = m(\alpha) + \sum_{i=1}^p a_i M_i + \sum_{i=p+1}^n a_i m(\mu_i).$$

So the coefficient of t^n for large n is

$$A(n - M) h_p(0, \ldots, 0, \mu_{p+1}, \ldots, \mu_n) \prod_{j=p+1}^n |\mu_j|^{z + b_j - \beta a_j - 1},$$

and the propositions are proved.

2. Preliminary constructions. The initial, but not the critical, steps in the construction of the Igusa data attached to orbital integrals can be carried out in general. We begin with a reductive group G, a quasisplit group G^*, an isomorphism $\psi : G \to G^*$, and a distinguished pair $\mathbf{T}^* \subseteq \mathbf{B}^*$, where $\mathbf{B}^*$ is a Borel and $\mathbf{T}^*$ a Cartan subgroup of G^*. The groups $G, G^*, \mathbf{T}^*, \mathbf{B}^*$ are all to be defined over F, but the isomorphism ψ need only be defined over $\bar{F}$. In addition we suppose we are given a Cartan subgroup T of G and a Cartan subgroup T^* of G^*, both defined over F, as well as a commutative diagram

(2.0)
$$\begin{tikzcd} \mathbf{T}^* & T^* \arrow[l, "\eta_*"'] \\ & T \arrow[u, "\psi_{T,T^*}"'] \arrow[ul, "\eta"] \end{tikzcd}$$

which is the restriction of
$$\begin{tikzcd} G^* & G^* \arrow[l, "\eta_*"'] \\ & G \arrow[u, "\psi_{T,T^*}"'] \arrow[ul, "\eta"] \end{tikzcd}$$

The homomorphism η_* is to be inner, and ψ_{T,T^*} is to be of the form ad $h \circ \psi$, with $h \in G^*(\bar{F})$. Moreover the restriction of ψ_{T,T^*} to T is to be defined over F. Since the diagram (2.0) will be fixed we may replace ψ by ad $h \circ \psi$ and suppose that $\psi = \psi_{T,T^*}$. The isomorphism ψ allows us to identify G and G^* when the F-structure is not being considered.

The isomorphism η allows us to identify the Weyl group of T and $\mathbf{T}^*$, both of which we label Ω, and to introduce the Borel subgroup $\mathbf{B} = \eta^{-1}(\mathbf{B}^*)$ of G. The Weyl chambers in $X^*(T) \otimes \mathbf{R}$, the module $X^*(T)$ being the group of rational characters, may be labelled by $\omega \in \Omega$. We set $W(\omega) = \omega^{-1} W_+$ if W_+ is the chamber positive with respect to $\mathbf{B}$, and if $W = W(\omega)$ we set $\mathbf{B}(W) = \mathbf{B}^\omega = w^{-1}\mathbf{B}w$, w being a representative of ω in the normalizer of T. We introduce an action of the Galois group on the set $\mathfrak{w}$ of Weyl chambers by

$$\sigma(\mathbf{B}(W)) = \mathbf{B}(\sigma_T(W)), \qquad \sigma \in \text{ Gal}(\bar{F}/F).$$

Let V be the variety of Borel subgroups of G. It is defined over F although it may have no F-valued points. We call a point $(B(W) \mid W \in \mathfrak{w})$ in $V^{\mathfrak{w}}$ a *regular star* if there exists an $h \in G$ such that

$$B(W) = h^{-1}\mathbf{B}(W)h = \mathbf{B}(W)^h$$

for all W. The variety S^0 of regular stars is a locally closed subvariety of $V^{\mathfrak{w}}$.

If α is a simple root of T in $\mathbf{B}$ with associated reflection $\epsilon = \epsilon(\alpha)$ let $\mathbf{P}_\alpha$ be the smallest parabolic subgroup containing $\mathbf{B}$ and $\mathbf{B}^\epsilon$. The point $(B(W) \mid W \in \mathfrak{w})$ is called a *star* if for every ω and every α there is an h such that

$$B(W(\omega)) = h^{-1}\mathbf{B}h$$

and

$$hB(W(\epsilon\omega))h^{-1} \subset \mathbf{P}_\alpha.$$

The variety S of stars is closed.

The group G acts to the right on S^0 and S:

$$g : (B(W)) \to (B(W)^g) = (g^{-1}B(W)g).$$

The group Ω acts to the left

$$\mu : (B(W)) \to (B'(W))$$

with $B'(W(\omega)) = B(W(\omega\mu))$.

As subvarieties of $V^{\mathfrak{w}}$ the varieties S^0 and S are naturally defined over F, but this is not the F-structure we want. We define the Galois action on $S(\bar{F})$ and $S^0(\bar{F})$ by

$$\sigma : (B(W)) \to (\sigma(B(\sigma_T^{-1}(W)))).$$

2.1. Lemma. *(a) There are natural bijections between the following three sets: (i) $T(\bar{F})\backslash G(\bar{F})$; (ii) $S^0(\bar{F})$; (iii) the collection of pairs (T', ν) where T' is a Cartan subgroup of G over $\bar{F}$ and $\nu : T \to T'$ is induced by an inner automorphism of $G(\bar{F})$.*

(b) If the class $T(\bar{F})h, h \in G(\bar{F})$, corresponds to $e \in S^0(\bar{F})$ and to (T', ν) then the following conditions are equivalent: (i) $h \in \mathfrak{a}(T/F)$; (ii) $e \in S^0(F)$; (iii) T' and ν are defined over F.

The bijections whose existence is asserted in (a) are given by $h \to (B(W) = \mathbf{B}(W)^h)$ and $h \to (T', \nu)$ with $T' = T^h, \nu(t) = t^h = h^{-1}th$. The only doubtful point is the equivalence of (ii) in (b) with (i) and (iii). However

$$\sigma(B(\sigma_T^{-1}W)) = (\sigma(\mathbf{B}(\sigma_T^{-1}W)))^{\sigma(h)} = \mathbf{B}(W)^{\sigma(h)}$$

and

$$\mathbf{B}(W)^{\sigma(h)} = \mathbf{B}(W)^h$$

for all W if and only if $\sigma(h)h^{-1} \in T(\bar{F})$.

We next introduce the variety consisting of all points (g, e) in $G \times S$ such that $g \in B(W)$ for all W, the point e being $(B(W))$. Let X^0 be the open subvariety of pairs (g, e) with g and e regular and let X be its closure. Both G and Ω act on X and X^0, the element h in G taking (g, e) to $(h^{-1}gh, e^h)$ and $\mu \in \Omega$ taking it to $(g, \mu(e))$. Let π be the morphism $(g, e) \to g$ from X to G. It is proper. There is in addition clearly a unique morphism $\varphi : X \to T$ such that for a given $x = (g, e)$ there is some h in G satisfying

$$B(W_+) = h^{-1}\mathbf{B}h \tag{2.2}$$

and

$$\varphi(x) \equiv hgh^{-1}(\mathrm{mod}\ \mathbf{N}), \tag{2.3}$$

where $\mathbf{N}$ is the unipotent radical of $\mathbf{B}$.

2.4. Lemma. *(a) We have*

$$\varphi(\mu(x)) = \mu(\varphi(x)).$$

(b) The morphism φ is defined over F.

Both assertions need only be verified on X^0. Multiplying the h appearing in (2.2) and (2.3) on the left by an element of $\mathbf{B}$ does not affect either equation. Thus we may suppose, provided that x is in X^0, that

$$B(W) = \mathbf{B}(W)^h \tag{2.5}$$

for all W. Then $g = h^{-1}\varphi(x)h$. Passing to $\mu(x)$ replaces $B(W(\omega))$ by $B(W(\omega\mu))$ which is equal to $\mathbf{B}(W)^{mh}$, where m is a representative of μ. Thus h is replaced by mh and $\varphi(x)$ by $m\varphi(x)m^{-1} = \mu(\varphi(x))$.

If $x = (g, e)$ lies in $X^0(\bar{F})$ then passing to $\sigma(x)$ we replace g by $\sigma(g)$ and h in (2.5) by $\sigma(g)$. Thus hgh^{-1} is replaced by $\sigma(hgh^{-1})$ and $\varphi(x)$ by $\sigma(\varphi(x))$.

The next assertion is an immediate consequence of the basic theory of reductive algebraic groups.

2.6. Lemma. *If $x = (g, e)$ lies in X and g is regular and semi-simple then $x \in X^0$.*

If γ in $T(F)$ is regular let $\varphi_\gamma^{-1}(F) = \{x \in X^0(F) \mid \varphi(x) = \gamma\}$. Let Ω_γ be the stabilizer of γ in Ω and Ω_γ^0 the subgroup of elements in Ω_γ fixed by $\sigma_T, \sigma \in \text{Gal}(\bar{F}/F), \sigma_T$ denoting the natural action of σ on the Weyl group of T.

2.7. Lemma. *The morphism π defines a bijection of the orbits of Ω_γ^0 in $\varphi_\gamma^{-1}(F)$ with the stable conjugacy class of γ in $G(F)$.*

The stable conjugacy class is by definition ([5]) given by

$$\{a^{-1}\gamma a \mid a \in \mathfrak{A}(T/F)\}$$

and thus is the quotient $G_\gamma(\bar{F}) \cap \mathfrak{A}(T/F)\backslash\mathfrak{A}(T/F)$. Moreover

$$T(\bar{F})\backslash G_\gamma(\bar{F}) \cap \mathfrak{A}(T/F) = \Omega_\gamma^0.$$

On the other hand if $a \in \mathfrak{A}(T/F)$ then $(a^{-1}\gamma a, (\mathbf{B}(W)^a))$ lies in $\varphi_\gamma^{-1}(F)$. It is also clear that if $x = (a^{-1}\gamma a, (\mathbf{B}(W)^a))$ lies in $X^0(F)$ then $a \in \mathfrak{A}(T/F)$. The lemma follows readily.

The variety Y^0 occurring in the Igusa data will be obtained by taking a smooth curve C in T which passes through the origin s_0 where its tangent is regular in the sense that it does not lie in a hyperplane defined by a root and which contains no other singular point. Then Y^0 will be the inverse image of $C^0 = C - \{s_0\}$ in X^0. Since X^0 is clearly isomorphic over F to $T^0 \times T\backslash G$, where T^0 is the set of regular elements in T and $(\gamma, g) \to (g^{-1}\gamma g, (\mathbf{B}(W)^g))$ the morphism φ is smooth on X^0 and Y^0 is smooth. Before carrying out the construction of Y itself, which is at the moment possible only in very simple cases, we introduce a form on X^0 from which the Igusa form will be derived.

Let M be the variety providing the Grothendieck-Springer resolution. Its points are $\{(g, B) \in G \times V \mid g \in B\}$ and it is smooth. We have a morphism $\xi : (g, e) \to (g, B(W_+))$ from X to M. The variety M may be identified with a quotient $\mathbf{B} \times_{\mathbf{B}} G$ by means of the morphism

$$(b, g) \to (g^{-1}bg, g^{-1}\mathbf{B}g)$$

(cf. [1]).

2.8. Lemma. *Let $\{\mu_i\}$ be a basis for the left-invariant 1-forms on $\mathbf{B}$ and $\{\omega_j\}$ a basis for the right-invariant forms on G which vanish on the Lie algebra of $\mathbf{B}$. For each i let ν_i be a right-invariant form on G whose restriction to $\mathbf{B}$ is equal to μ_i at the identity. Then the form on $\mathbf{B} \times G$ which at $a \times g$ is given by*

$$(\wedge_i(\mu_i, (1 - \text{ ad } a)\nu_i)) \wedge (\wedge_j(0, \omega_j)) \tag{2.9}$$

is the pull-back to $\mathbf{B} \times G$ of a nowhere vanishing form ω_M on M of maximal degree. The form ω_M is G-invariant.

The final statement is clear. Otherwise we must verify that (2.9) is $\mathbf{B}$-invariant and that its contraction with the tangent vector to a curve $(\exp tXa\exp(-tX), \exp tXg)$, X in the Lie algebra of $\mathbf{B}$, is zero at $t = 0$. Since the tangent vector is $((\text{ad } a^{-1} - 1)X, X)$, the first factor being regarded as a left-invariant vector field on $\mathbf{B}$ and the second as a right-invariant vector field on G the contraction is certainly zero.

The action of $b \in \mathbf{B}$ pulls the form $(\mu_i, (1 - \text{ ad}(bab^{-1}))\nu_i)$ at (bab^{-1}, bg) back to $(\text{ad } b^{-1}(\mu_i)$, $(1 - \text{ ad } a)\text{ad } b^{-1}(\nu_i))$ at (a, g) and ω_j back to $\text{ad } b^{-1}\omega_j$. Since

$$\det(\text{ad } b \mid_{\text{Lie } \mathbf{B}})\det(\text{ad } b \mid_{\text{Lie } \mathbf{B}\backslash\text{Lie } G}) = \det(\text{ad } b \mid_{\text{Lie} G}) = 1$$

the invariance is clear.

The basis $\{\omega_j\}$ together with the collection $\{\nu_i\}$ defines a basis for the dual of Lie G and thus a form ω_G of maximal degree on G. A suitable choice of bases leads to any given form. Consequently we may in particular assume that ω_G is defined over F. Suppose in addition that we also have an invariant form ω_T of maximal degree on T and that it is defined over F. Thus the quotient of ω_G by ω_T is a form $\omega_{T\backslash G}$ on $T\backslash G$.

The set of F-valued points of $T\backslash G$ is precisely the image of $\mathfrak{A}(T/F)$ in $T(\bar{F})\backslash G(\bar{F})$. If $\gamma \in T(F)$ then $\kappa(\epsilon)f(a^{-1}\gamma a)$, where $\epsilon = \epsilon(a)$ is a function on $(T\backslash G)(F)$ that the following lemma, which in characteristic zero is contained in Lemma 8.2 of [5] and whose proof we otherwise leave to the reader, shows to be locally constant.

2.10. Lemma. *If $\{\epsilon_\sigma\}$ is a 1-cocycle of Gal$(\bar{F}/F)$ with values in $T(\bar{F})$ and if all ϵ_σ are sufficiently close to 1 then it bounds.*

The κ-orbital integral $\Phi^\kappa(\gamma, f)$ is thus equal to

$$\int_{(T\backslash G)(F)} \kappa(\epsilon) f(a^{-1}\gamma a)|\omega_{T\backslash G}|. \tag{2.11}$$

However the morphism $a \to (a^{-1}\gamma a, (\mathbf{B}(a)^a))$ is an isomorphism of $T\backslash G$ with $\varphi_\gamma^{-1}(F)$ and it is defined over F. Thus we may regard $\omega_{T\backslash G}$ as a form on and (2.11) as an integral over $\varphi^{-1}(F)$.

On the other hand if ω_{X^0} is a form of maximal degree on X^0 then the quotient $\frac{\omega_{X^0}}{\omega_T}$ defines a form ω_γ on each fibre.

2.12. Lemma. *The form*

$$\omega_{X^0} = \xi^* \omega_M$$

is defined over $\bar{F}$ and

$$\omega_\gamma = (\prod_{\alpha>0} 1 - \alpha^{-1}(\gamma))\omega_{T\backslash G}.$$

For the proof we may identify X^0 with $T^0 \times T\backslash G$ and $\omega_{T\backslash G}$ is the quotient $\frac{\omega_T \wedge \omega_{T\backslash G}}{\omega_T}$. We choose a basis $\{X_i\}$ of Lie T such that $\omega_T(\wedge_i X_i) = 1$, extend it to a basis $\{X_i\} \cup \{Y_i\}$ of Lie $\mathbf{B}$, and finally to a basis $\{X_i\} \cup \{Y_i\} \cup \{Z_i\}$ of Lie G which we may assume is dual to $\{\nu_i\} \cup \{\omega_i\}$. Since ξ is defined by the natural morphism $T \times T\backslash G = T \times_T G \to \mathbf{B} \times_{\mathbf{B}} G$ it follows readily from (2.9) that

$$\xi^* \omega_M = \prod_{\alpha>0} (1 - \alpha^{-1}(\gamma))\omega_T \wedge \omega_{T\backslash G},$$

an equality which yields the second half of the lemma. The first is clear. Moreover

$$\sigma \frac{(\Pi_{\alpha>0} 1 - \alpha^{-1}(\gamma))}{\Pi_{\alpha>0} 1 - \alpha^{-1}(\gamma)} = \lambda_\sigma(\gamma),$$

$\sigma \in$ Gal$(\bar{F}/F)$, vanishes nowhere and $\sigma(\omega_{X_0}) = \lambda_\sigma(\gamma)\omega_{X_0}$.

As a consequence, rather than studying the integral $\Phi^\kappa(\gamma, f)$ we prefer to study

$$\begin{aligned}(\prod_{\alpha>0} |1 - \alpha^{-1}(\gamma)|)\Phi^\kappa(\gamma, f) &= F^\kappa(\gamma, f) \\ &= \int_{\varphi_\gamma^{-1}(F)} m_\kappa(e(x)) f(g(x))|\omega_\gamma|\end{aligned}$$

where $m_\kappa(e) = \kappa(\epsilon(a))$ if e in $S^0(F)$ corresponds to a in $\mathfrak{A}(T/F)$.

We define the Igusa form ω^0 on Y^0 by choosing coordinates $\lambda_1, \dots, \lambda_\ell$ at 1 on T such that C is defined by $\lambda_2 = \cdots = \lambda_n = 0$ and λ_1 restricted to C gives the coordinate λ. We suppose moreover that each is defined over F and that $\omega_T = \beta d\lambda_1 \wedge \cdots \wedge d\lambda_\ell$. We choose ω' such that

$$\omega_{X^0} = \beta d\lambda_2 \wedge \cdots \wedge d\lambda_\ell \wedge \omega'$$

and let ω^0 be the restriction of ω' to Y^0. We take the Igusa integrand f_κ to be the restriction $m_\kappa(e(x))f(g(x))$ to Y^0 and obtain

$$F^\kappa(\gamma, f) = \int_{Y_{s(\lambda)}} f_\kappa(y)|\omega_\lambda|, \qquad \gamma \in C^0.$$

3. Some useful coordinates and a first resolution. We shall construct the variety Y in two stages, beginning with a resolution S_1 of S, and to this end we introduce some affine coordinate patches on S. Each patch will be associated to a fixed Borel subgroup B_∞ and a Borel subgroup B_0 in opposition to it, and will be defined over F if B_∞ and B_0 are, but this we do not assume. Let N_∞ be the unipotent radical of B_∞. It is an affine space and the morphism $n \to n^{-1}B_0 n$ is an isomorphism of N_∞ with the open subvariety $V(B_\infty) \subseteq V$ of Borel subgroups in opposition to B_∞. Let $S(B_\infty) \subseteq S$ be formed by those families $(B(W))$ such that $B(W)$ is in opposition to B_∞ for all W and let $S^0(B_\infty) = S^0 \cap S(B_\infty)$. If

$$S(B_\infty, B_0) = \{(B(W)) \in S(B_\infty) \mid B(W_0) = B_0\}$$

then $(B(W), n) \to B(W)^n$ is an isomorphism of $S(B_\infty, B_0) \times N_\infty$ with $S(B_\infty)$. We have as well

$$S^0(B_\infty) \simeq S^0(B_\infty, B_0) \times N_\infty.$$

We fix our attention on $S(B_\infty, B_0)$, setting $T_0 = B_0 \cap B_\infty$. The inclusion $T_0 \subseteq B_0$ allows us to identify Ω with the Weyl group of T. If (W, β) is a pair consisting of a Weyl chamber and a wall of it, or more precisely a root defining a wall, then for some unique ω and some simple root $\alpha = \alpha(W, \beta)$, we have $W = W(\omega), \beta = \alpha \circ \omega$.

We shall attach to each such pair a coordinate function $z(W, \beta)$ on $S(B_\infty, B_0)$. To this purpose we fix for each simple root α two root vectors X_α and $X_{-\alpha}$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$ with $\beta(H_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ for all roots β. Let G_α be the group whose Lie algebra is spanned by $X_\alpha, X_{-\alpha}, H_\alpha$.

Now let ϵ be the reflection corresponding to $\alpha(W, \beta)$, so that $\beta = 0$ is the wall separating $W(\omega)$ from $W(\epsilon\omega) = W(\omega')$. Thus if $e = (B(W))$ lies in $S(B_\infty, B_0)$ then for some h in N_∞, we have

$$B(W(\omega)) = B_0^h \text{ and } h(B(W(\omega')))h^{-1} \subseteq G_\alpha B_0.$$

As a consequence

$$hB(W(\omega'))h^{-1} = \exp(-zX_{-\alpha})B_0 \exp(zX_{-\alpha}).$$

The coefficient z is uniquely determined and we set $z(W, \beta) = z$. Observe that $z(W, \beta) = 0$ if and only if $B(W(\omega)) = B(W(\omega'))$.

3.1. Lemma. *The coordinates $z(W,\beta)$ are regular functions on $S(B_\infty, B_0)$ and determine the point e.*

More formally stated the lemma would read:

$$S(B_\infty, B_0) \simeq \text{Spec}(K(z(W,\beta))),$$

K being a field over which B_∞ and B_0 are defined.

To prove it we introduce *paths*, which are sequences $W_0, W_1, \ldots, W_p$ of Weyl chambers such that for each i the chambers W_i and W_{i+1} are distinct but have a wall, $\beta_{i+1} = 0$, in common. Let $W_i = W(\omega_i)$ and set $\alpha_{i+1} = \omega_i(\beta_{i+1})$, so that

$$\omega_i = \epsilon(\alpha_i) \cdots \epsilon(\alpha_1)\omega_0,$$

where $\epsilon(\alpha_i)$ is the reflection associated to α_i. It is clear that if $hB(W_0)h^{-1} = B_0$, with $h \in N_\infty$ and $z_j = z(W_{j-1}, \beta_j), \alpha_j = \alpha(W_{j-1}, \beta_j)$ then

$$hB(W_i)h^{-1} = \exp(-z_1 X_{-\alpha_1}) \cdots \exp(-z_i X_{-\alpha_i}) B_0 (\exp z_i X_{-\alpha_i}) \cdots \exp(z_1 X_{-\alpha_1}).$$

Taking W_0 to be W_+ we see that the point is determined by the coordinates. We see moreover that if C is a closed path, so that $W_0 = W_p$, then

$$\exp(z_p X_{-\alpha_p}) \cdots \exp(z_1 X_{-\alpha_1}) = 1.$$

These are polynomial relations among the coordinates and form a complete set of relations. One particularly simple type of relation results from taking a closed path of length two. If W and W' are adjacent and separated by the wall $\beta = 0$ then

$$z(W', \beta) = -z(W, \beta).$$

In general if a star is given then the sequence $B(W_0), \ldots, B(W_p)$ attached to a path is a gallery in the Tits building attached to G. It follows readily from [7], especially the section 3.4, that the star is regular if and only if $B(W') \neq B(W)$ for all pairs of adjacent chambers W, W'. In particular a star in $S(B_\infty, B_0)$ lies in $S^0(B_\infty, B_0)$ if and only if all its coordinates are non-zero.

The torus T_0 acts on $S(B_\infty, B_0)$ and on $S^0(B_\infty, B_0)$ by sending $e = (B(W))$ to $e^t = (B(W)^t)$. Clearly the coordinates $z'(W,\beta)$ of e^t are given by

$$z'(W(\omega), \alpha \circ \omega) = \alpha(t) z(W(\omega), \alpha \circ \omega).$$

Let $S'(B_\infty, B_0)$ be the subvariety of $e \in S(B_\infty, B_0)$ such that for each simple root α there is at least one chamber $W(\omega)$ for which $z(W(\omega), \alpha \circ \omega) \neq 0$. The quotient $R(B_\infty, B_0)$ of $S'(B_\infty, B_0)$ by T_0 exists and $S'(B_\infty, B_0)$ is a principal bundle over $R(B_\infty, B_0)$ with group $(T_0)_{\mathrm{ad}}$, the image of T_0 in G_{ad}. Let A^Σ be the product over the set of simple roots of the one-dimensional coordinate space over F. The group T_0 acts on A^Σ by

$$t : x = (x_\alpha) \to x^t = (\alpha^{-1}(t)x_\alpha).$$

Set

$$S_1(B_\infty, B_0) = S'(B_\infty, B_0) \times_{T_0} A^\Sigma$$

and

$$S_1(B_\infty) = S_1(B_\infty, B_0) \times N_\infty.$$

The morphism $e \to e \times (1, \dots, 1)$ imbeds S' as an open subvariety of S_1.

There is a lemma to be verified.

3.2. Lemma. *(a) $E(B_\infty) \otimes_\sigma \bar{F} \simeq \sigma(S(B_\infty)) = S(\sigma(B_\infty))$.*

(b) $S'(B_\infty) \otimes_\sigma \bar{F} \simeq \sigma(S'(B_\infty)) = S'(\sigma(B_\infty))$.

(c) The isomorphism $S'(B_\infty) \otimes_\sigma \bar{F} \simeq S'(\sigma(B_\infty))$ extends to

$$S_1(B_\infty) \otimes_\sigma \bar{F} \simeq S_1(\sigma(B_\infty)).$$

The isomorphisms in (a) and (b) are formal, $S(B_\infty)$ and $S'(B_\infty)$ both being contained in S, which we have defined as a variety over F by twisting the natural F-structure. The equality of (a) is also immediately clear, but not that of (b).

Suppose $e \times n$ lies in $S(B_\infty, B_0) \times N_\infty$ and $e = (B(W))$ with

$$B(W) = B_0^{n_W n}.$$

Here

$$n_W = \exp(z_p X_{-\alpha_p}) \cdots \exp(z_1 X_{-\alpha_1}),$$

where $z_1, \dots, z_p$ are the coordinates attached to a path from W_+ to W. Represent $S(\sigma(B_\infty))$ as $S(\sigma(B_\infty), \sigma(B_0)) \times \sigma(N_\infty)$. Then $\sigma \in \mathrm{Gal}(\bar{F}/F)$ takes $e \times n$ to $e' \times n'$ with

$$n' = \sigma(n_{\sigma_T^{-1}(W_+)} n) \tag{3.3}$$

and $e' = (B'(W))$,

$$B'(W)^{n'} = \sigma(B(\sigma_T^{-1}(W))^{n'} = \sigma(B_0)^{\sigma(n_{\sigma_T^{-1}(W)} n)}. \tag{3.4}$$

Thus if for simplicity we take $X_{-\sigma(\alpha)} = \sigma(X_{-\alpha})$ then

$$z'(W, \beta) = \sigma(z(\sigma_T^{-1}(W), \sigma_T^{-1}(\beta))),$$

and the equality of (b) follows.

On $S_1(B_\infty, B_0)$ we define the elements n_W by

$$n_W = \exp(x_p z_p X_{-\alpha_p}) \cdots \exp(x_1 z_1 X_{-\alpha_1})$$

where $x_i = x_{\alpha_i}$ if $e_1 = e \times (x_\alpha)$ and the α_i are associated to the path as before. The action on the simple roots defined by $\mathbf{T}^*$ is denoted $\sigma_{\mathbf{T}^*}$ and σ_T and $\sigma_{\mathbf{T}^*}$ differ by an element of the Weyl group. Consequently

$$\sigma_{\mathbf{T}^*}^{-1}(\alpha(W,\beta)) = \alpha(\sigma_T^{-1}(W), \sigma_T^{-1}(\beta))$$

for all W and β. The action of $\sigma \in \operatorname{Gal}(\bar{F}/F)$ on $S_1(B_\infty, B_0) \times N_\infty$ which yields the isomorphism of (c) can then be taken to be

$$\sigma : e \times (x_\alpha) \times n \to e' \times (\sigma(x_{\sigma_{\mathbf{T}^*}^{-1}(\alpha)})) \times n',$$

where e' and n' are defined by (3.3) and (3.4).

Although it is of no importance for us we note that the following lemma can be proved in exactly the same way.

3.5. Lemma. *The subvariety $S'(B_\infty)$ is invariant under the action of Ω and this action extends to $S_1(B_\infty)$.*

We also note that the dependence of $S_1(B_\infty)$ on the choise of B_0 is only apparent.

3.6. Lemma. *Any two choices of a Borel subgroup in opposition to B_∞ lead to canonically isomorphic varieties $S_1(B_\infty)$.*

Let B_0 and $B_0' = B_0^h, h \in N_\infty$ be two choices. Then $T_0' = T_0^h$ and we may take $X_{-\alpha'} = X_{-\alpha}^h, \alpha'$ being the root defined by $\alpha'(t^h) = \alpha(t)$. The isomorphism between $S(B_\infty, B_0') \times N_\infty$ and $S(B_\infty, B_0) \times N_\infty$ defined by identifying them both with $S(B_\infty)$ takes $e \times n$ to $e' \times n'$ where $n = hn'$ and

$$z'(W,\beta) = z(W,\beta).$$

Recall that the relations $T_0 \subseteq B_0, T_0^h \subseteq B_0^h, \mathbf{T} \subseteq \mathbf{B}$ allow us to identify the root systems of all three tori.

We now take two Borel subgroups B_∞ and B_∞' and construct a birational map from $S_1(B_\infty)$ to $S_1(B_\infty')$. These will allow us to paste together the varieties $S_1(B_\infty)$ to form a variety S_1.

We first make the birational map from $S(B_\infty)$ to $S(B_\infty')$ defined by the imbeddings $S(B_\infty) \subseteq S, S(B_\infty') \subseteq S$ explicit. We represent both varieties as products $S(B_\infty, B_0) \times N_\infty, S(B_\infty', B_0) \times N_\infty$ with respect to a Borel subgroup B_0 opposite to them both. If $e \times n$ is equal to $e' \times n'$ then n' is defined by the condition $n \in B_0 n'$. Let e be given by $(B(W))$ and e' by $(B'(W))$, so that

$$B(W)^n = B'(W)^{n'}.$$

If, for a given $W, B'(W) = B_0^{h'}, h' \in N'_\infty$, and $B(W) = B_0^h, h \in N_\infty$, then $bh = h'n'n^{-1}, b \in B_0$. Thus if W' is adjacent to W and separated from it by the wall $\beta = 0$, so that

$$B'(W') = B^{(\exp z' X_{-\alpha'})h'}$$

then

$$B(W) = B^{\exp(z'X_{-\alpha'})bW} = B_0^{\exp(z'X^b_{-\alpha'})h}.$$

Now $-\alpha'$ is the negative of the root $\alpha = \alpha(W, \beta)$, the prime indicating that it is a root of $T'_0 = B'_\infty \cap B_0$ in B'_∞. There is clearly a linear fractional transformation taking 0 to 0, and thus of the form

$$\frac{1}{z'} = \frac{a}{z} + c$$

with $a \neq 0$, such that

$$\exp z' X^b_{-\alpha'} \in B_0 \exp z X_{-\alpha'}.$$

This z is $z(W, \beta)$. The two coefficients $a = a(W, \beta)$ and $c = c(W, \beta)$ depend on e but only through b, and thus only on the coordinates of a path from W to W_+.

We define the birational map between $S_1(B_\infty)$ and $S_1(B'_\infty)$ by

(3.7) $$x'_{\alpha'} = x_\alpha$$

and

(3.7) $$z(W, \beta) = \alpha(W, \beta) z'(W, \beta) + c(W, \beta) z(W, \beta) z'(W, \beta) x_\alpha.$$

The first equation allows us to rewrite the second as

$$\frac{1}{x'_\alpha z'(W, \beta)} = \frac{a(W, \beta)}{x_\alpha z(W, \beta)} + c(W, \beta).$$

This makes it clear that the map takes equivalent points to equivalent points. It defines an isomorphism between the following two open subvarieties of $S_1(B_\infty)$ and $S_1(B'_\infty)$:

(i) $\{e, (x_\alpha), n) \mid n_W n \in B_0 N'_\infty \ \forall\ W\}$

(ii) $\{(e', (x'_{\alpha'}), n') \mid n'_W n' \in B_0 N_\infty \ \forall\ W\}$.

We can use these isomorphisms to paste $S_1(B_\infty)$ and $S_1(B'_\infty)$ together. Examining the process on the dense subvarieties $S'(B_\infty), S'(B'_\infty)$ we see that the pasting is well-defined and independent of the choice of B_0 and consistent, and thus defines a pre-variety S_1.

3.9. Lemma. *The pre-variety S_1 is a variety defined over F on which G and Ω act and there is a morphism $p : S_1 \to S$ defined over F and compatible with the actions of G and Ω. The morphism p is a birational map between S_1 and the closure of S^0 in S.*

Since we have patched along sub-varieties that were as large as possible the Hausdorff axiom is easily verified. Lemma 3.2 allows us to define it over F, and the actions of G and Ω are clear. The morphism p is defined locally and sends a point $e \times (x_\alpha) \times n$ in $S'(B_\infty, B_0) \times A^\Sigma \times N_\infty$ to $(B(W)) = (B_0^{nwn})$.

In general one has to expect that S_1 is only a first approximation to a smooth completion of S^0. However for the simple cases with which this paper is ultimately concerned we have the following lemma.

3.10. Lemma. (a) *If the Dynkin diagram is of type A_1 then $S_1 = S = \mathbf{P}^1 \times \mathbf{P}^1$ is smooth and complete.*

(b) *If the Dynkin diagram is of type A_2 then S_1 is smooth and complete and p is surjective.*

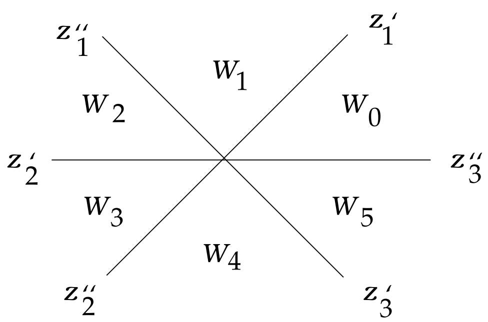
The first part of the lemma is of course immediate and is included only as a reference. We begin the proof of the second with some simple remarks applicable to all groups of rank two.

Fix B_∞ and B_0 and let α', α'' be the two simple roots of T_0 in B_0. Let $W_0 = W_+, W_1, \ldots, W_{2r-1}$ be the path C which starts at W_+, crosses $\alpha' = 0$, and then continues until it returns to W_+ through $\alpha'' = 0$. The corresponding coordinates will be denoted $z_1', z_1'', z_2', z_2'', \ldots, z_r', z_r''$. In addition we denote $X_{-\alpha'}$ by X' and $X_{-\alpha''}$ by X''. Then the sole equation to be satisfied is

$$\exp z_r'' X'' \exp z_r' X' \cdots \exp z_1'' X'' \exp z_1' X' = 1, \tag{3.11}$$

which when examined is seen to consist of r polynomial relations among the $2r$ coordinates. It can be given various forms by taking inverses or by permuting the coordinates cyclically.

It is convenient to refer to a diagram of chambers and walls labelled by the corresponding coordinates. For types A_2 this is



Two equations which follow immediately from (3.11) are:

$$\sum_{i=1}^{r} z_i' = 0; \sum_{i=1}^{r} z_i'' = 0. \tag{3.12}$$

Thus on $S'(B_\infty, B_0)$ at least two of the z'_i and two of the z''_i are not zero. We now take $r = 3$ and for convenience read the indices modulo 3. We work on $S'(B_\infty, B_0)$ and distinguish two possibilities:

(a) There is an i such that $z'_i \neq 0, z''_i \neq 0, z'_{i+1} \neq 0$.

(b) If $z'_i \neq 0, z'_{i+1} \neq 0$ then $z''_i = 0$.

If (b) obtains then $z''_i \neq 0, z''_{i+1} \neq 0$ imply $z'_{i+1} = 0$. Indeed either $z'_i \neq 0$ or $z'_{i+2} \neq 0$. Thus if $z'_{i+1} \neq 0$ either (z'_i, z''_i, z'_{i+1}) or $(z'_{i+1}, z''_{i+1}, z'_{i+2})$ contradict (b). If (a) obtains then all coordinates are non-zero. This, like many other things, can best be seen in the Tits building. The condition (a) is that there is no folding at three consecutive stages in the gallery associated to the path C and the family $(B(W))$ and thus no folding at all.

For many purposes, in particular for the proof of the lemma, we may replace a point in $S(B_\infty, B_0)$ by a translate under the Weyl group. Thus we suppose that either (A): for all i both z'_i and z''_i are different from zero or (B): $z''_1 = 0, z'_3 = 0$ but all other coordinates are non-zero.

To prove the first part of the lemma we need only show that $S'(B_\infty, B_0)$ is smooth at such a point. We have a morphism m from $2r$-dimensional affine space to N_∞ given by

$$m : (z', z'') \to \exp z''_r X'' \exp z'_r X' \cdots \exp z''_1 X'' \exp z'_1 X'$$

and it is enough to show that its Jacobian is of rank r at such a point. Set $V'_i = \exp(-z'_i X'), V''_i = \exp(-z''_i X'')$. The map on tangent spaces is given explicitly by:

$$\begin{aligned}
&\frac{\partial}{\partial z'_1} \to X'; \quad \frac{\partial}{\partial z''_1} \to \text{ ad } V'_1(X'');\\
&\frac{\partial}{\partial z'_2} \to \text{ ad } V'_1 \text{ ad } V''_1(X') = \text{ ad } V'_1 \text{ ad } V''_1 \text{ ad } V'_2(X');\\
&\frac{\partial}{\partial z''_2} \to \text{ ad } V'_1 \text{ ad } V''_1 \text{ ad } V'_2(X'');\\
&\frac{\partial}{\partial z'_3} \to \text{ ad } V'_1 \text{ ad } V''_1 \text{ ad } V'_2 \text{ ad } V''_2(X').
\end{aligned}$$

There is a similar formula for the image of $\frac{\partial}{\partial z''_3}$ but it is not necessary, for the images of $\frac{\partial}{\partial z'_2}, \frac{\partial}{\partial z''_2}, \frac{\partial}{\partial z'_3}$ are already linearly independent, and thus span the Lie algebra of N_∞. This is clear because X', X'' and

$$\text{ad } V''_2(X') = X' - z''_2[X'', X']$$

are linearly independent.

To complete the proof of the lemma we need only show that p takes $S_1(B_\infty, B_0)$ onto $S(B_\infty, B_0)$ and that the full inverse image of $S(B_\infty, B_0)$ is $S_1(B_\infty, B_0)$, for $S_1(B_\infty, B_0)$ is then the blow-up of $S(B_\infty, B_0)$ with respect to the ideal generated by $\{z'_i z''_j\}$ and thus proper over it.

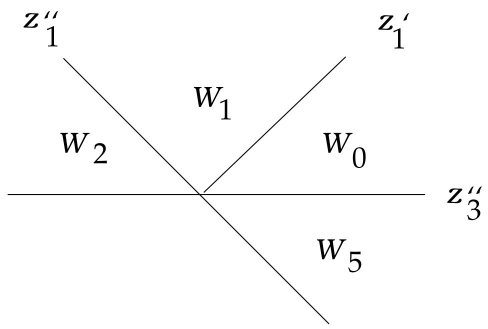
That $S'(B_\infty, B_0) \times A^\Sigma \to S(B_\infty, B_0)$ is surjective is tantamount to the claim that if $x_i', x_i'', 1 \le i \le 3$ are given with

$$\exp x_3'' X'' \exp x_3' X' \cdots \exp x_1'' X'' \exp x_1' X' = 1$$

then we can find μ', μ'' and $z_i', z_i'', 1 \le i \le 3$ such that at least one of the z_i' and at least one of the z_i'' are not zero, equation (3.11) is satisfied, with of course $r = 3$, and finally $x_i' = \mu' z_i', x_i'' = \mu'' z_i''$ for all i.

This is clear if at least one x_i' and at least one x_i'' are not zero, or if all the coordinates are zero. Therefore suppose, for example, that all of the x_i' are zero but that at least one of the x_i'' is not zero. If one of the x_i'' is zero, we take it, with no loss of generality to be x_1''. Then $x_2'' = -x_3''$ and we set $z_i'' = x_i''$ for all i and $z_3' = 0, z_1' = z, z_2' = -z, z \ne 0$.

If none of the x_i'' is zero choose $z_1'' = x_1'', z_3'' = x_3'', z_1' \ne 0$ and form the configuration of Borel subgroups specified by the following diagram of Weyl chambers and coordinates:



The Borel subgroups $B(W_2)$ and $B(W_5)$ will necessarily be in opposition (cf. [7]). Thus the diagram can be completed by the addition of $B(W_3)$ and $B(W_4)$. If they are both in opposition to B_∞ we will have a point of $S'(B_\infty, B_0)$ and z_2'' will, as a consequence of (3.12), be equal to x_2''.

In fact to complete the diagram we choose $z_2'' = x_2''$ and solve the equation

$$\exp(-z_2' X') \exp(-z_2'' X'') \exp(-z_3' X') = \exp(z_1'' X'') \exp(z_1' X') \exp(z_3'' X'')$$

for z_2' and z_3'. Setting $z_3' = z, z_2' = -z_1' - z$ we find that this is equivalent to

$$\exp zX' A \exp(-zX') = B$$

with

$$A = \exp(z_1' X') \exp(-z_2'' X''), \qquad B = \exp(z_1'' X'') \exp(z_1' X') \exp(z_3'' X'').$$

Since A is congruent to B modulo the derived group of N_∞ and $z_2'' \ne 0$ this is possible.

It must finally be verified that if $e \times (\mu', \mu'')$ in $S'(B_\infty', B_0) \times A^\Sigma$ maps to a point in $S(B_\infty, B_0)$ then it is equivalent to a point in $S'(B_\infty, B_0) \times A^\Sigma$. This is however an easy consequence of (3.7) and (3.8). For example the image of $S'(B_\infty, B_0) \times_T \{0\}$ in $S_1(B_\infty, B_0)$ is independent in B_∞ and isomorphic to $R(B_\infty, B_0)$.

There are two observations that it will be useful to record now for use in the next paragraph. We have introduced two types (a) and (b) of points in $S'(B_\infty, B_0)$.

3.13. Lemma. *(i) For a point of type (a) all coordinates z'_i and z''_i are non-zero.*

(ii) For a point of type (b) exactly one coordinate z'_{i_0} and one coordinate z''_{j_0} are non-zero. Moreover $j_0 \equiv i_0 + 1 (\mathrm{mod}\, 3)$, so that the corresponding walls are collinear.

This we have already seen. It means that there are four types A, and B_1, B_2, B_3. For type B_i the coordinate z''_i is 0.

3.14. Lemma. *If $e \times (x_{\alpha'}, x_{\alpha''}) \times n$ in $S'(B_\infty, B_0) \times A^\Sigma \times N_\infty$ and $e' \times (x'_{\alpha'}, x'_{\alpha''}) \times n'$ in $S'(B'_\infty, B'_0) \times A^\Sigma \times N'_\infty$ have the same image in S_1 then e and e' are of the same type.*

This follows readily from (3.8), which shows that $z'(W, \beta) = 0$ if and only if $z(W, \beta) = 0$, and allows us to assign a type to points in S_1.

4. The final desingularization. Let X_1 be the closure of X^0 in $G \times S_1$, which of course contains $G^0 \times S^0$ as an open subvariety, G^0 being the set of regular semi-simple elements in G. Let Y_1 be the closure of Y^0 in X_1. Then Y^0 will be an open subvariety of Y_1. We will modify Y_1 along a sub-variety of $Y_1 - Y^0$ to obtain Y.

We begin by covering Y_1 with open subvarieties whose structure can be examined in detail. We can project Y_1 to S_1. Let $Y_1(B_\infty)$ and $Y_1(B_\infty, B_0)$ be the inverse images of $S_1(B_\infty)$ and $S_1(B_\infty, B_0)$. Then $Y_1(B_\infty)$ is open and $Y_1(B_\infty) \simeq Y_1(B_\infty, B) \times N_\infty$. It is also easily seen that $Y_1(B_\infty, B_0)$ is the closure of its intersection with Y^0. There is an inner automorphism taking the pair T_0, B_0 to $T, \mathbf{B}$. Let C_0 be the inverse image of C. We also use λ as a parameter on C_0.

A point y_0 in $Y_1(B_\infty, B_0)$ has the form $(s(\lambda_0)n_0, e_0), s(\lambda_0) \in C_0, n_0 \in N_0, e_0 \in E_1(B_\infty, B_0)$. We consider the equations satisfied by a point $y = (s(\lambda)n, e)$ in $Y^0 \cap Y_1(B_\infty, B_0)$ near y_0. Let $p(e) = (B(W))$. Let $Y_1^j(B_\infty, B_0), j = 1, 2, 3$ be the open subvariety of points of type A or B_j and $Y_1^j(B_\infty) = Y_1^j(B_\infty, B_0) \times N_\infty$. Since the varieties $Y_1^j(B_\infty)$ are permuted amongst themselves by the Weyl group, it suffices to consider $Y_1^1(B_\infty, B_0)$. The proof of Lemma 3.10 shows that we may take z'_1, z''_1, z''_3 as coordinates on $S'(B_\infty, B_0)$ near e_0. Since $z''_3 \neq 0$ at e_0 we may take $x = \mu' z'_1, y = -\mu'' z''_3$, and $V = -\frac{z'_1}{z''_3}$ as coordinates on $S_1(B_\infty, B_0)$ near e_0.

On S_0 we use λ as coordinate and we specify coordinates u, v, w for $n \in N_0$ by the relation

$$n = \exp u X_{\alpha'} \exp v X_{\alpha''} \exp w [X_{\alpha'}, X_{\alpha''}].$$

Since $B(W_2)$ and $B(W_5)$ are in opposition when y lies in Y^0 the conditions $s(\lambda)n \in B(W_i), 0 \leq i < 6$, reduce to $s(\lambda)n \in B(W_i)$ for $i = 1, 2, 5$. They can be used to solve for u, v, and w.

Observing that

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}$$

is upper triangular if and only if $xu = 1 - \beta^{-1}$ we find, for example, that

$$xu = \lambda b(\lambda) \tag{4.1}$$

with $\lambda b(\lambda) = 1 - \alpha'(s(\lambda)^{-1})$. The other equations are

$$yv = \lambda c(\lambda) \tag{4.2}$$

with $\lambda c(\lambda) = 1 - \alpha''(s(\lambda)^{-1})$, and

$$xyVw = \lambda(d(\lambda) - Vc(\lambda))$$

where $\lambda d(\lambda) = 1 - \alpha'(s(\lambda)^{-1})\alpha''(s(\lambda)^{-1})$. Thus $b(\lambda), c(\lambda), d(\lambda)$ are all regular and invertible at $\lambda = 0$ and $d(0) \neq c(0)$, for it has been assumed that the tangent to C lies in no singular hyperplane. Set $U = d(\lambda) - Vc(\lambda)$ and rewrite the final equation as

$$xyVw = \lambda U. \tag{4.3}$$

It is convenient to consider the two regions

$$Y_1^{11}(B_\infty, B_0) = \{y \in Y_1^1(B_\infty, B_0) \mid U \neq 0\}$$

and

$$Y_1^{12}(B_\infty, B_0) = \{y \in Y_1^1(B_\infty, B_0) \mid V \neq 0\}$$

separately. They clearly cover $Y_1^1(B_\infty, B_0)$, at least in a neighborhood of $\lambda = 0$.

4.4. Lemma. *$Y_1^{11}(B_\infty, B_0)$ is non-singular.*

We may choose x, y, V, and w as coordinates.

4.5. Lemma. *$Y_1^{12}(B_\infty, B_0)$ is non-singular except at $x = y = U = u = v = w = 0$.*

We deduce from (4.1), (4.2), (4.3) the following equations on $Y_1^{12}(B_\infty, B_0) \cap Y^0$:

$$xu = M_1 yv, \quad xw = M_2 vU, \quad yw = M_3 uU, \tag{4.6}$$

in which M_1, M_2, M_3 are nowhere vanishing functions and $M_2 = M_1 M_3$. Replacing the coordinates u and v by $u' = M_3 u$ and $v' = M_2 v$ one simplifies the equations and verifies the lemma readily, showing of course at the same time that $Y_1^{12}(B_\infty, B_0)$ is defined by the equations (4.6) and either of the expressions (4.1) or (4.2) for λ.

If the group is an inner form of $SL(3)$ we can now define Y by taking $y^{11}(B_\infty, B_0) = Y_1^{11}(B_\infty, B_0), y^{11}(B_\infty) = Y^{11}(B_\infty, B_0) \times N_\infty$, and by taking $Y^{12}(B_\infty, B_0)$ to be the result of blowing up $Y_1^{12}(B_\infty, B_0)$ at $x = y = U = u = v = w = 0, Y^{12}(B_\infty)$ to be $Y^{12}(B_\infty, B_0) \times N_\infty$, and then pasting. Since we are in effect simply blowing up along the singular subvariety of Y_1 the pasting is possible and yields, thanks to the universal property of blowing up ([3], Proposition 7.14), a smooth variety Y defined over F. The composition of $Y \to Y_1 \hookrightarrow X_1 \to X$ with $\pi : X \to G$ or $\varphi : X \to T$ yields morphisms $\pi : Y \to G$ and $\varphi : Y \to C$.

4.6. Proposition. *All three of Y, π, and φ are defined over F. The morphism $\pi \times \varphi$ is proper and φ is an Igusa fibering.*

We have only to verify the last point. At the same time we see what further modification is necessary when G is an outer form, and in addition note various properties of φ.

On $Y^{11}(B_\infty)$ the morphism φ is defined by

$$\lambda = \alpha xyVw,$$

where α here and below denotes a regular function that vanishes nowhere on the region under consideration. Thus there are four divisors of $\lambda = 0 : E_1'(x = 0); E_1''(y = 0); E_2^1(V = 0); E_3(w = 0)$. The regions $Y^{21}(B_\infty), Y^{31}(B_\infty)$ yield in addition divisors E_2^2, E_2^3.

On $Y_1^{12}(B_\infty) = Y_1^{22}(B_\infty) = Y_1^{32}(B_\infty) = Y_1^{12}(B_\infty, B_0) \times N_\infty$ we use the coordinates x, y, U, u', v', w. On $Y^{12}(B_\infty)$ we take one of these to be 1 and multiply them all by an additional coordinate t. We examine several domains separately, giving the expression for φ and labelling the divisors of $\lambda = 0$, two divisors having the same label only if they overlap.

Fixed variable	Independent coordinates	λ	Divisors
(i) $x = 1$	y, v', U, t	$\alpha t^2 yv'$	$E_1''(y = 0), E_3(v' = 0), E_5(t = 0)$
(ii) $y = 1$	x, u', U, t	$\alpha t^2 xu'$	$E_1'(x = 0), E_3(u' = 0), E_5$
(iii) $u' = 1$	y, v', w, t	$\alpha t^2 yv'$	$E_1'(v' = 0), E_4(y = 0), E_5$
(iv) $v' = 1$	x, u', w, t	$\alpha t^2 xu'$	$E_1''(u' = 0), E_4(x = 0), E_5$
(v) $w = 1$	u', v', U, t	$\alpha t^2 u'v'U$	$E_1', E_1'', E_4(U = 0), E_5$
(vi) $U = 1$	x, y, w, t	$\alpha t^2 xyw$	$E_1', E_1'', E_3(w = 0), E_5$

A variant of Hilbert's Theorem 90 shows that a divisor through a point over F is locally defined by a function over F if and only if it is fixed by the Galois group. Thus to verify that $Y \to C$ is an Igusa fibering one need only verify that any two divisors in $\varphi^{-1}(C_0)$ with a point in common lie in different orbits under the Galois group. If the form is inner then an orbit is contained in one of the sets $\{E_1'\}, \{E_1''\}, \{E_2^1, E_2^2, E_2^3\}, \{E_3\}, \{E_4\}, \{E_5\}$. Since the $E_2^i, i = 1, 2, 3$ are mutually disjoint this condition is clearly satisfied. However if the form is outer then E_1' and E_1'' form a single orbit, and a further blowing up is necessary in order to separate them.

Thus we blow up along the variety which on each of $Y^{j1}(B_\infty)$ is given by $x = y = 0$, does not meet the domains (i)-(iv), in (v) is given by $u' = v' = 0$, and in (vi) by $x = y = 0$. The proposition (4.6) is still satisfied but one more divisor E_6 is added to $\varphi^{-1}(C_0)$. The exponent $a_6 = a(E_6)$ is clearly 2, and the others appear above.

4.7. *The exponents $a(E), E \in \mathfrak{E}$, are $a_1' = a_1'' = a_2^1 = a_2^2 = a_2^3 = a_3 = a_4 = 1, a_5 = a_6 = 2$.*

To define the form ω^0 on Y^0 we introduced the variety M. There is of course a morphism $M \to T$ defined in the same way as φ. Let N be the inverse image of C. Then ξ takes Y^0 into N and may be extended to $\xi : Y \to N$. We choose $\lambda_1, \dots, \lambda_\ell$ as at the end of section 2 and let ω_N be the restriction to N of a form ω'' such that $\omega_M = \beta d\lambda_2 \wedge \cdots \wedge d\lambda_\ell \wedge \omega''$. Then

$$\omega = \xi^* \omega_N$$

is an extension of ω to Y. It will be an Igusa form.

4.8. *The exponents $b(E), E \in \mathfrak{E}$, are $b_1' = b_1'' = 2, b_2^1 = b_2^2 = b_2^3 = 3, b_3 = 4, b_4 = 1, b_5 = 5, b_6 = 4$.*

The exponents can be calculated on $Y^{ij}(B_\infty, B_0)$ and are those of the form $\xi^*(d\lambda \wedge du \wedge d\nu \wedge dw)$. On $Y^{11}(B_\infty, B_0)$ they are determined by the Jacobian

$$\frac{\partial(\lambda, u, \nu, w)}{\partial(x, y, V, w)}.$$

Since we are only interested in the divisor of this function we may replace V by $V' = \frac{V}{U}$, and are led to

$$\frac{\partial(xyV'w, A(xyV'w)yV'w, B(xyV'y)xV'w, w)}{\partial(x, y, V', w)}$$
$$= AB\frac{\partial(xyV'w, yV'w, xV'w, w)}{\partial(x, y, V', w)} = ABxyV^2w^3.$$

This gives b_1', b_1'', b_2^1, and b_3. The other exponents are evaluated in a similar fashion.

The universality of blowing up implies that the groups G and Ω act on Y_0. If $E \in \mathfrak{E}$ let $\widehat{E}$ be the complement in E of its intersection with the other divisors. The morphism π maps $\widehat{E}$ onto a unipotent orbit in G, of which there are three, the regular, the semi-regular, and that of the identity.

4.9. *The divisors $\widehat{E}_3$ and $\widehat{E}_5$ map to the identity, $\widehat{E}_1', \widehat{E}_1'', \widehat{E}_2^1, \widehat{E}_2^2, \widehat{E}_2^3$, and $\widehat{E}_6$ to the semi-regular orbit, and $\widehat{E}_4$ to the regular orbit.*

5. The Igusa integrand. That the support of f_κ is proper over $C(F)$ follows immediately because f has compact support and the morphism from Y to $G \times C$ is proper. Moreover $f(g(y))$ is clearly locally constant. Thus the only point to verify is that $m_\kappa(e(y))$ has locally the form

$$m_\kappa(e(y)) = \gamma \sum_{i=1}^{n} \kappa_i(\mu_i), \tag{5.1}$$

where κ_i is trivial if $\mu_i = 0$ is not a divisor of $\varphi^{-1}(s_0)$ and is constant along divisors. Parts of the argument are quite general and with these we begin.

We first take G to be quasi-split, so that $G = G^*, T = T^*$, and $\eta = \eta_*$, and we choose two opposed Borel subgroups B_0 and B_∞ defined over F. Since the pair $(\mathbf{T}^*, \mathbf{B}^*)$ is conjugate under $G(F)$ to (T_0, B_0) we may suppose $\mathbf{T}^* = T_0, \mathbf{B}^* = B_0$. To each point e in $S^0(F)$ we may attach, according to Lemma 2.1, an element a in $\mathfrak{A}(T/F)$ and thus a cocycle $\{\epsilon_\sigma\}$ of Gal$(\bar{F}/F)$ with values in $T(\bar{F})$. Our first goal is to express $\{\epsilon_\sigma\}$ on $S^0(B_\infty) = S^0(B_\infty, B_0) \times N_\infty$ in terms of the coordinates $z(W, \beta)$.

We choose h in $G(\bar{F})$ so that ν is $g \to h^{-1}gh$ and set $w_\sigma = \sigma(h)h^{-1}, \sigma \in$ Gal$(\bar{F}/F)$. It lies in the normalizer of T. In order to define the coordinates $z(W, \beta)$ we have had to choose for each simple root α of T_0 in B_0 root vectors $X_\alpha, X_{-\alpha}$, and thus $H_\alpha = [X_\alpha, X_{-\alpha}]$. They define a subgroup G_α and an isogeny $\xi_\alpha : SL(2) \to G_\alpha$. Let w_α be the image of

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in G. Let $\alpha^\vee$ be the coroot corresponding to the root α and if $z \neq 0$ lies in a field containing F define $z^{\alpha^\vee}$ in T by

$$\mu(z^{\alpha^\vee}) = z^{\langle \mu, \alpha^\vee \rangle},$$

for all characters μ.

5.2. Proposition. *Let ω_- be the element of the Weyl group taking positive roots to negative roots and let w_- be a representative of it in the normalizer of T_0. Set $W_- = \omega_-(W_+)$. If $\sigma \in$ Gal$(\bar{F}/F)$ let ω_σ be the image of w_σ in the Weyl group and let $\epsilon(\alpha_j) \cdots \epsilon(\alpha_1)$ be a reduced expression for $\omega_-\omega_\sigma^{-1}\omega_-^{-1}$ as a product of reflections associated to simple roots. Let $\sigma(X_{-\alpha}) = \sigma_{T_0}(X_{-\alpha}) = u_\alpha X_{-\sigma(\alpha)}, \sigma^{-1}(\alpha_i) = \alpha_i'$, and let*

$$z_\kappa = z(W(\epsilon(\alpha'_{\kappa-1}) \cdots \epsilon(\alpha'_1)\omega_-\omega_{\sigma^{-1}}^{-1}, \omega_{\sigma^{-1}}\omega_-\epsilon(\alpha'_1) \cdots \epsilon(\alpha'_{\kappa-1})\alpha'_\kappa)).$$

Then

$$\sigma \to w_\sigma h \sigma(w_-^{-1}) w_{\alpha_j}(\sigma(z_j)u_{\alpha_j})^{\alpha_j^\vee} \cdots w_{\alpha_1}(\sigma(z_1)u_{\alpha_1})^{\alpha_1^\vee} w_- h^{-1}$$

is a cocycle of Gal$(\bar{F}/F)$ *with values in $T(\bar{F})$ and its class is that of $\{\epsilon_\sigma\}$.*

We will begin the proof with a lemma. Let $e = (B(W))$. We choose ν and $n_W, W \in \mathfrak{w}$, such that

$$B(W_+) = B_0^\nu, \quad B(W) = B_0^{n_W \nu}.$$

If for simplicity we write $\sigma_T(W) = \sigma(W)$ we have

$$\sigma(B_0^{n_W \nu}) = B_0^{\sigma(n_W \nu)} = B_0^{n_{\sigma(W)}\nu}$$

Thus $\sigma(n_W)\sigma(\nu)\nu^{-1}n_{\sigma(W)}^{-1}$ lies in B_0. Since it also lies in N_∞ it is 1. Consequently

$$\sigma(\nu)\nu^{-1} = \sigma(n_{\sigma^{-1}(W_+)}^{-1})$$

and

$$\sigma(n_{\sigma^{-1}(W)})\sigma(n^{-1}_{\sigma^{-1}(W_+)})n_W^{-1} = 1.$$

If $\omega = \epsilon(\alpha_p)\cdots\epsilon(\alpha_1)$ is a reduced expression for $\omega \in \Omega$ set

$$\omega_i = \epsilon(\alpha_i)\cdots\epsilon(\alpha_1), \quad \omega_0 = 1, \quad W_i = W(\omega_i), \quad \beta_i = \omega_{i-1}^{-1}\alpha_i,$$

and

$$z_i = z(W_{i-1}, \beta_i).$$

If $W = W(\omega)$ then

$$n_W = \exp(z_p X_{-\alpha_p})\cdots\exp(z_1 X_{-\alpha_1}).$$

Let N_ω be the connected subgroup of N_0, the unipotent radical of B_0, whose Lie algebra is spanned by $\{X_\alpha \mid \alpha > 0, \omega\alpha < 0\}$.

5.3. Lemma. *We have*

$$n_W \in N_0 w_{\alpha_p} z_{\alpha_p}^{\alpha_p^\vee}\cdots w_{\alpha_1} z_{\alpha_1}^{\alpha_1^\vee} N_\omega.$$

The equality

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} 1 & z^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\begin{pmatrix} 1 & z^{-1} \\ 0 & 1 \end{pmatrix}$$

shows that

$$\exp zX_{-\alpha} = \exp(z^{-1}X_\alpha)(w_\alpha z^{\alpha^\vee})\exp(z^{-1}X_\alpha).$$

This equality will allow us to verify the lemma by induction on p.

Let $\omega' = \epsilon\omega, \epsilon = \epsilon(\alpha)$, and suppose that $\omega' = \epsilon\epsilon(\alpha_p)\cdots\epsilon(\alpha_1)$ is a reduced expression for ω'. Let $R_\omega = \{\beta > 0 \mid \omega\beta < 0\}$. Then $\beta > 0$ lies in R_ω if and only if $\beta = 0$ separates $W(\omega)$ and W_+. Thus

$$R_{\omega'} = R_\omega \cup \{\omega^{-1}\alpha\}.$$

Set

$$w = w_{\alpha_p} z_{\alpha_p}^{\alpha_p^\vee}\cdots w_{\alpha_1} z_{\alpha_1}^{\alpha_1^\vee}$$

and suppose $n_W = uwu', u \in N_0, u' \in N_\omega$. If $W' = W(\omega')$ then

$$n_{W'} = \exp zX_{-\alpha} n_W$$

with $z = z(W, \omega^{-1}\alpha)$, and we must show that

$$w_\alpha z^{\alpha^\vee}\exp z^{-1}X_\alpha uw \in N_0 w_\alpha z^{\alpha^\vee} w N_\omega.$$

Let $N_0^\alpha \subseteq N_0$ be the connected subgroup of N_0 whose Lie algebra is spanned by $\{X_\beta \mid \beta > 0, \beta \neq \alpha\}$. Then for some $n \in N_0^\alpha$ and some y,

$$w_\alpha z^{\alpha^\vee} \exp z^{-1} X_\alpha u = w_\alpha z^{\alpha^\vee} n \exp y X_\alpha \in N_0 w_\alpha z^{\alpha^\vee} \exp y X_\alpha.$$

But for some x and some choice of $X_{\omega^{-1}(\alpha)}$

$$\exp y X_\alpha w = w \exp x X_{\omega^{-1}(\alpha)}.$$

This proves the lemma, and we return to the proposition.

If W_- the negative Weyl chamber and $n_{W_-} = u' w_- u$ with u, u' in N_0 then the torus $T(e) = \cap_W B(W)$ is

$$B(W_-) \cap B(W_+) = T^{hu\nu}$$

and the cocycle $\{\epsilon_\sigma\}$ is given by

$$\epsilon_\sigma = \sigma(hu\nu)\nu^{-1}u^{-1}h^{-1} = w_\sigma h(\sigma(u)\sigma(\nu)\nu^{-1}u^{-1})h^{-1}$$

Set

$$c_\sigma = (w_-^{-1}) w_{\alpha_j}(\sigma(z_j)u_{\alpha_j})^{\alpha_j^\vee} \cdots w_{\alpha_1}(\sigma(z_1)u_{\alpha_1})^{\alpha_1^\vee} W_-.$$

It is enough to show that for some $t \in T(\bar{F})$,

$$\sigma(u)\sigma(\nu)\nu^{-1}u^{-1} \in N_\infty c_\sigma N_\infty h^{-1}\sigma(t)t^{-1}h,$$

the proposition then following from the unicity of the Bruhat decomposition and the fact that $\epsilon_\sigma \in T(\bar{F})$.

We have

$$\sigma(u)\sigma(\nu)\nu^{-1}u^{-1} = \sigma(u)\sigma(n^{-1}_{\sigma^{-1}(W_+)})u^{-1}.$$

We use the relations $n_{W_-} = u' w_- u$ and

$$\sigma(n_{\sigma^{-1}(W_-)})\sigma(n^{-1}_{\sigma^{-1}(W_+)})n^{-1}_{W_-} = 1$$

to see that this lies in

$$N_\infty \sigma(w_-^{-1})\sigma(n_{W_-})\sigma(n^{-1}_{\sigma^{-1}(W_-)})w_- N_\infty.$$

The w_- which appears here is a representation of ω_- determined by n_{W_-} and thus by e, while the proposition is stated for an arbitrary representative. We shall show that it is valid for the particular representative chosen. However if we replace w_- by $w_- t, t \in T_0(\bar{F})$, then we multiply c_σ on the right by $\omega_\sigma(\sigma(t))t^{-1}$ or

$$h^{-1}(\sigma(hth^{-1})ht^{-1}h^{-1})h.$$

Observe that ω_σ is in the Weyl group of T_0 and not of T, and consequently represented by $h^{-1}(\sigma(h)h^{-1})h = h^{-1}\sigma(h)$. Thus if the proposition is valid for one choice of w_- it is valid for all.

Define $\mathbf{B}^*(W)$ as we defined $\mathbf{B}(W)$, but with $\mathbf{T}^*$ replacing T, and define an action of $\mathrm{Gal}(\bar{F}/F)$ on $\mathfrak{w}$ by

$$\sigma(\mathbf{B}^*(W)) = \mathbf{B}^*(\sigma^*(W)).$$

Then $\sigma^*(W_\pm) = W_\pm$. Since

$$\mathbf{B}^*(W) = \mathbf{B}(W)^h$$

we have

$$\sigma(W) = \omega_\sigma(\sigma^*(W)).$$

In particular

$$\sigma^{-1}(W_-) = \omega_{\sigma^{-1}} W_-.$$

Modifying the notation of the proposition we let $\epsilon(\alpha_i)\cdots\epsilon(\alpha_1)$ be a reduced expression for $\omega_-\omega_{\sigma^{-1}}^{-1}$ and $\epsilon(\alpha_m)\cdots\epsilon(\alpha_{i+1})$ be a reduced expression for $\omega_-\omega_{\sigma^{-1}}\omega_-^{-1}$. Then $\epsilon(\alpha_m)\cdots\epsilon(\alpha_1)$ is a reduced expression for ω_-. Consequently, with an obvious notation,

$$n_{W_-}n_{\sigma^{-1}(W_-)}^{-1} = \exp z_m X_{-\alpha_\nu}\cdots \exp z_{i+1}X_{-\alpha_{i+1}}$$

and

$$\sigma(n_{W_-}n_{\sigma^{-1}(W_-)}^{-1}) = \exp\sigma(z_m)\sigma(X_{-\alpha_m})\cdots\exp\sigma(z_{i+1})\sigma(X_{-\alpha_{i+1}}).$$

Since $\epsilon(\sigma(\alpha_m))\cdots\epsilon(\sigma(\alpha_{i+1}))$, where it is understood that $\sigma(\alpha_k) = \sigma_{T_0}(\alpha_k)$, is a reduced expression for $\sigma(\omega_-\omega_{\sigma^{-1}}\omega_-^{-1}) = \omega_-\omega_\sigma^{-1}\omega_-^{-1}$, the proposition follows from Lemma 5.3.

It is useful to supplement it by another. Let Θ be a subgroup of the quotient by T of the stabilizer of T in the group of automorphisms of G. Suppose that Θ is a semi-direct product $\Theta_2 \rtimes \Theta_1$ and that there is a set of roots $\{\beta_1,\cdots,\beta_r\}$ of T in G invariant under Θ_1 and such that Θ_2 is generated by the reflections $\epsilon_i = \epsilon(\beta_i)$ subject to the sole relations

$$(\epsilon_i\epsilon_j)^{m_{ij}} = 1.$$

Here $m_{ii} = 1$. However if $i \neq j$ then $\epsilon_k\epsilon_j$ acts as a rotation on the plane of $\beta_i^\vee, \beta_j^\vee$ and m_{ij} is its order.

5.4. Proposition. *Suppose that for each i we are given $x_i = x(\beta_i)$ in $F^\times$ and that $\beta_i \to x(\beta_i)$ is constant on orbits of Θ_1 in $\{\beta_1,\cdots,\beta_r\}$. There is then a unique cocycle δ of Θ with values in $T(\bar{F})$ such that*

$$\delta_\theta = 1, \quad \theta\in\Theta_1, \quad \textit{and} \quad \delta_{\epsilon_i} = x_i^{\beta_i^\vee}.$$

The cocycle is clearly unique and it is enough to show that it is well defined on Θ_2, which means that for each i and j

$$(x_i^{\beta_i^\vee}\epsilon_i(x_j^{\beta_j^\vee})\epsilon_i\epsilon_j(x_i^{\beta_i^\vee}\epsilon_i(x_j^{\beta_j^\vee}))\cdots(\epsilon_i\epsilon_j)^{m-1}(x_i^{\beta_i^\vee}\epsilon_i(x_j^{\beta_j^\vee}))=1.$$

If $i=j$ this is clear because

$$\epsilon_i(x^{\beta_i^\vee})=x^{-\beta_i^\vee}$$

for all x. If $i\neq j$ set $\epsilon=\epsilon_i\epsilon_j$. The product on the left is

$$x_i^\mu\epsilon_i(x_j)^\nu$$

with

$$\mu=\beta_i^\vee+\epsilon\beta_i^\vee+\cdots+\epsilon^{m-1}\beta_i^\vee$$

and

$$\nu=\beta_j^\vee+\epsilon\beta_j^\vee+\cdots+\epsilon^{m-1}\beta_j^\vee.$$

Since ϵ is a non-trivial rotation in the plane of $\{\beta_i^\vee,\beta_j^\vee\}$ both μ and ν are zero.

If $\theta=\theta_2\rtimes\theta_1$ and $\epsilon(\beta_{i_1})\cdots\epsilon(\beta_{i_j})$ is an expression for θ_2 as a product of reflections then

$$\delta_\theta=\epsilon(\beta_{i_1})\cdots\epsilon(\beta_{i_{j-1}})(x_{i_j}^{\beta_{i_j}^\vee})\epsilon(\beta_{i_1})\cdots\epsilon(\beta_{i_{j-2}})(x_{i_{j-1}}^{\beta_{i_{j-1}}^\vee})\cdots\epsilon(\beta_{i_1})(x_{i_2}^{\beta_{i_2}^\vee})x_{i_1}^{\beta_{i_1}^\vee}.$$

If $\Theta\supseteq\{\omega_-\sigma_T\omega_-^{-1}\mid\sigma\in\operatorname{Gal}(\bar F/F)\}$ then $\sigma\to\omega_-(\delta_{\omega_-\sigma_T\omega_-^{-1}})$ defines a cocycle of $\operatorname{Gal}(\bar F/F)$ with values in $T(\bar F)$. Consequently, if we denote by s the number of Θ_1-orbits in $\{\beta_1,\cdots,\beta_r\}$, the proposition provides us with a continuous homomorphism δ of $(F^\times)^s$ into $H^1(F,T)$, indeed into $\mathcal{E}(T/F)$, and allows us to associate to any character κ of this group one of $(F^\times)^s$.

Suppose for example that for each simple root α we choose $x_\alpha\in F^\times$ and in the cocycle of Proposition 5.2 we multiply z_j by x_{α_j}. This multiplies the value of the cocycle at $\sigma=w_\sigma\times\sigma^*$ by

$$h(\omega_-(\epsilon(\alpha_1)\cdots\epsilon(\alpha_{j-1})(x_{\alpha_j}^{\alpha_j^\vee})\epsilon(\alpha_1)\cdots\epsilon(\alpha_{j-2})(x_{\alpha_{j-1}}^{\alpha_{j-1}^\vee})\cdots x_{\alpha_1}^{\alpha_1^\vee}))h^{-1}.$$

Since $\epsilon(\alpha_1)\cdots\epsilon(\alpha_j)$ is a reduced expression for $\omega_-\omega_\sigma\omega_-^{-1}$, the cocycle itself is multiplied by $\delta(\{x_\alpha\})$, defined provided that $x_\alpha=x_{\sigma^*(\alpha)}$ for all α and σ. As h is given, we have used it to identify roots of T and T_0.

In this example we have taken $\Theta_2=\Omega$. Similar considerations apply to any Θ_2. The collection of roots $\Delta'=\{\omega\beta_i\mid 1\le i\le r,\omega\in\Theta_2\}$, the Θ_2-roots, allow us to introduce a new decomposition of $X^*(T)\otimes\mathbf{R}$ into chambers W', on which Θ_2 acts simply transitively. These chambers will have walls $\gamma'=0$, and to each pair (W',γ') we can associate one $\beta_i=\beta'(W',\gamma')$, where $W'=\omega^{-1}W'_+,\gamma'=\beta_i\circ\omega$. Here $\omega\in\Theta_2$ and W'_+ is the chamber defined by $\beta_j>0,1\le j\le r$. Suppose that $W_+\subseteq W'_+$. A wall (W,γ) for Ω will be said to be a Θ_2 wall if $\gamma\in\Delta'$. Then W is contained in a chamber W' of which $\gamma=0$ is a wall. Set $\beta'(W,\gamma)=\beta'(W',\gamma)$.

If $\omega = \epsilon(\beta_{i_1})\cdots\epsilon(\beta_{i_j})$ is a reduced expression for ω as an element of Θ_2 then a reduced expression for it in Ω is obtained simply by substituting reduced expressions for the $\epsilon(\beta_{i_k})$. The only Θ_2-hyperplane separating $\epsilon(\beta_{i_k})W_+$ from W_+ is $\beta_{i_k} = 0$. Thus the only Θ_2-hyperplane separating $\epsilon(\beta_{i_1})\cdots\epsilon(\beta_{i_k})W_+$ from $\epsilon(\beta_{i_1})\cdots\epsilon(\beta_{i_{k-1}})W_+$ is

$$\epsilon(\beta_{i_1})\cdots\epsilon(\beta_{i_{k-1}})\beta_{i_k} = 0.$$

Consequently if $\omega = \epsilon(\alpha_1)\cdots\epsilon(\alpha_\ell)$ is the reduced expression for ω obtained from the reduced expressions for the $\epsilon(\beta_{i_k})$ then the sequence $\beta_{i_1}, \epsilon(\beta_{i_1})\beta_{i_2}, \ldots, \epsilon(\beta_{i_1})\cdots\epsilon(\beta_{i_{j-1}})\beta_{i_j}$ is just the subsequence of Θ_2-roots in $\alpha_1, \epsilon(\alpha_1)\alpha_1, \ldots, \epsilon(\alpha_1)\cdots\epsilon(\alpha_{\ell-1})\alpha_\ell$.

Supposing then that Θ_2 contains $\{\omega_-\sigma_T\omega_-^{-1} \mid \sigma \in \mathrm{Gal}(\bar{F}/F)\}$ and that the $x_i, 1 \le i \le r$, are chosen as in Proposition 5.4 we multiply the z_k appearing in the cocycle of Proposition 5.2 by x_i each time that $\epsilon(\alpha_1)\cdots\epsilon(\alpha_{k-1})\alpha_k$ is a Θ_2-wall of $\epsilon(\alpha_1)\cdots\epsilon(\alpha_{k-1})W_+$ with $\beta'(\epsilon(\alpha_1)\cdots\epsilon(\alpha_{k-1})W_+, \epsilon(\alpha_1)\cdots\epsilon(\alpha_{k-1})\alpha_k) = \beta_i$. The effect is to multiply its class by $\delta(\{x_i\})$.

Referring to Proposition 5.2 we observe that

$$\omega_{\sigma^{-1}}\omega_-(\epsilon(\alpha'_1)\cdots\epsilon(\alpha'_{k-1})\alpha'_k) = \omega_-(\omega_-\omega_{\sigma^{-1}}\sigma^{-1}\omega_-^{-1}(\epsilon(\alpha_1)\cdots\epsilon(\alpha_{k-1})\alpha_k)).$$

Since the collection of Θ_2-roots is invariant under Θ and $\omega_-\omega_{\sigma^{-1}}\sigma^{-1}\omega^{-1} \in \Theta$ we conclude that $\epsilon(\alpha_1)\cdots\epsilon(\alpha_{k-1})\alpha_k$ is a Θ_2-root if and only if $\omega_-^{-1}\omega_{\sigma^{-1}}\omega_-(\epsilon(\alpha'_1)\cdots\epsilon(\alpha'_{k-1})\alpha'_k)$ is.

We can now verify for a quasi-split group that $m_\kappa(e(y))$ has the form (5.1) in a neighborhood of a point y_0 in Y. We may work on $Y^{11}(B_\infty, B_0)$ or on one of the regions into which $Y^{12}(B_\infty, B_0)$ has been divided, the groups B_∞, B_0 being defined over F. The admissible coordinates may differ from the coordinates used previously, for they are not necessarily defined over F. However any admissible coordinate defining a divisor in $\mathfrak{E}$ will be the product of one of these with a regular function which does not vanish in the neighborhood.

On $Y^{12}(B_\infty, B_0)$ all points are of type A. Thus $z(W,\beta)$ will be equal to $x\bar{z}(W,\beta)$ or $y\bar{z}(W,\beta)$, according to $\alpha(W,\beta)$ is α' or α'', where $\bar{z}(W,\beta)$ is regular and does not vanish at $e(y_0)$. The class of the cocycle of Proposition 5.2 depends locally only on the coordinates x and y. On the individual regions we have (i) $x = \bar{x}, y = \nu\bar{y}$, (ii) $x = \mu\bar{x}, y = \bar{y}$, (iii) $x = \nu\eta\bar{x}, y = \nu\bar{y}$, (iv) $x = \mu\bar{x}, y = \mu\eta\bar{y}$, (v) $x = \mu\xi\bar{x}, y = \nu\xi\bar{y}$, (vi) $x = \mu\bar{x}, y = \nu\bar{y}$, where μ, ν, η, ξ denote admissible local coordinates and $\bar{x}, \bar{y}$ are regular and do not vanish in the neighborhood and thus do not affect the local behavior of $m_\kappa(e)$.

When we blow up $\mu = \nu = 0$ in (v) and (vi) to obtain $Y^{12}(B_\infty, B_0)$ for an outer form then E'_1 and E''_1 cease to have F-valued points and near an F-valued point on E_6 we can write $x = \mu\bar{x}, y = \mu\bar{y}$ or $x = \mu\xi\bar{x}, y = \mu\xi\bar{y}$.

We see immediately that $m_\kappa(e)$ will be locally the product of a constant function and one of $\kappa(\delta(1,\nu)), \kappa(\delta(\mu,1)), \kappa(\delta(\nu\eta,\nu)$ or

$$\kappa(\delta(\mu\xi,\mu\xi)),$$

Θ_2 being taken to be Ω and Θ_1 the image of $\mathrm{Gal}(\bar{F}/F)$ in the group of outer automorphisms.

Near a point of type B we have to consider smaller Θ_2. Take for example the region $Y^{11}(B_\infty, B_0)$. We will have $x = \mu\bar{x}, y = \nu\bar{y}$ (or $x = \mu\bar{x}, y = \mu\bar{y}$ if the form is outer) and $V = \xi\bar{V}$. Apart from a constant factor, $m_\kappa(e)$ will be the product of $\kappa(\delta(\mu,\nu))$ (or $\kappa(\delta(\mu,\mu))$ with Θ as before and a factor associated to a smaller Θ. Since the point is assumed to be rational and of type B_1 the Galois group must leave the line through the walls associated to z_1'', z_3' invariant. Let β be the root associated to this wall and let $\Theta_2 = \{1, \epsilon(\omega_-\beta)\}$. Let Θ_1 be the group of automorphisms fixing $\omega_-\beta$. Then $\Theta \supseteq \{\omega_-\sigma_T\omega_-^{-1}\}$. Applying ω_- to the walls to which z_1', z_2', z_2'', z_3'' are attached does not yield a Θ_2-wall and $z_i' = \mu\bar{z}_i', i = 1, 2, z_i'' = \nu\bar{z}_i'', i = 2, 3$. However ω_- applied to the walls to which z_3' and z_1'' are attached does yield a Θ_2-wall and $z_1' = \mu\xi\bar{z}_1', z_1'' = \nu\xi\bar{z}_1''$ (or $\mu\xi\bar{z}_1''$ for an outer form). The factor attached to the smaller Θ is $\kappa(\delta(\xi))$.

Since a form of $SL(3)$ is either quasi-split or anisotropic it remains to verify (5.1) for an anisotropic group. For this we use a lemma which follows readily from the construction of Y.

5.5. Lemma. *If $y_0 \in Y$ and $p(e(y_0)) = (B(W))$ then*

$$\dim \cap_W (B(W) \geq 2.$$

If y_0 is an F-valued point then $\cap_W B(W)$ is defined over F. Thus if the group G is anistropic it must be a Cartan subgroup T', for it is solvable. To each Weyl chamber W' in $X^*(T') \otimes \mathbf{R}$ is associated a Borel subgroup $B(W')$ and there is a map $\eta : W \to \eta(W) = W'$ such that $B(W) = B(\eta(W))$. If W and W_1 are adjacent then either $\eta(W)$ and $\eta(W_1)$ are adjacent or $\eta(W) = \eta(W_1)$. Hence the collection $\{\eta(W) \mid W \in \mathfrak{w}\}$ forms a connected family of Weyl chambers lying in no half-plane, for otherwise $\cap B(W)$ would properly contain T'. We conclude that η is bijective, that $e(y_0)$ lies in S^0, and hence that $m_\kappa(e)$ is constant in a neighborhood of $e(y_0)$.

References

[1] E. Brieskorn, *Singular elements of semi-simple algebraic groups*, Proc. Int. Cong. of Mathematicians, Nice (1970).

[2] Harish-Chandra, *A formula for semi-simple Lie groups*, Amer. Jour. Math. LXXIX (1957).

[3] R. Hartshorne, *Algebraic Geometry*, Springer Verlag (1977).

[4] J.-I. Igusa, *Lectures on forms of higher degree*, Tata Institute of Fundamental Research, Bombay (1978).

[5] R.P. Langlands, *Les débuts d'une formule des traces stable*, Publ. Math. de l'Univ. de Paris VII, **13** (1983).

[6] D. Shelstad, *Orbital integrals, endoscopic groups and L-indistinguishability for real groups*, Publ. Math. de l'Univ. de Paris VII, to appear.

[7] J. Tits, *Buildings of spherical type and finite BN-pairs*, Lect. Notes in Math. **386** (1974).